International Doctorate in Civil and Environmental Engineering

Anisotropic Structures - Theory and Design

Strutture anisotrope: teoria e progetto

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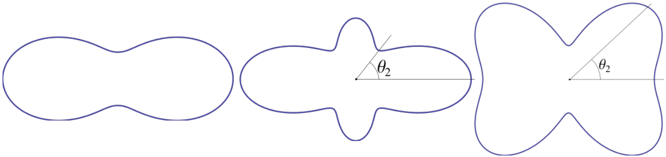

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Topics of the fourth lesson

- The Polar Formalism - Part 1

Why the polar formalism?

In 1979 G. Verchery presented a memory about the invariants of an elasticity-type tensor. This short paper marks the birth of the polar formalism or method.

We have seen that for anisotropic materials the Cartesian components of a tensor describing a given property all depend upon the direction; moreover, this dependence is rather cumbersome.

Hence, when the Cartesian components are used for representing an anisotropic tensor, none of these components are an intrinsic quantity: all of them are frame-dependent parameters.

Here intrinsic is only a synonymous of invariant but it has also a more physical signification: it indicates a quantity that characterizes intrinsically a physical property, that belongs, in some sense, to it.

In addition, if a priviledged direction linked to the anisotropic property exist, it does not appear explicitly.

On its side, the polar formalism is an algebraic technique to represent a plane tensor using only tensor invariants and angles (that is why the method is polar).

Hence, the intrinsic quantities describing a given anisotropic property and the direction directly and explicitly appear in the equations.

It is exactly the use of invariants and angles that makes the polar method interesting for analyzing anisotropic phenomena:

- the invariants are not linked to the particular choice of the axes, so they give an intrinsic representation of elasticity
- the explicit use of angles makes appear directly one of the fundamental aspects of anisotropy: the direction. This is possible because, unlike other tensor representations, the polar method does not use exclusively polynomial invariants

- the invariants used in the polar formalism are linked to the elastic symmetries, i.e. they represent in an invariant way the symmetries
- the polar method allows for obtaining much simpler formulae for the rotation of the axes than the Cartesian ones
- the method is based upon the use of a special complex variable transformation, that is why it can be used only for representing plane tensors
- for its characteristics, the polar formalism is well suited for design problems and for theoretical analyses. The possibility of working directly with tensor invariants gives in fact some mathematical advantages in certain transformations.

An algebraic approach to elastic symmetries?

Because the polar invariants represent intrinsically the symmetries, the polar formalism opens the way to a new approach to the analysis of the material symmetries. While in a traditional approach the analysis of the symmetries is essentially geometric, in the polar formalism it is strictly algebraic.

In fact, with the traditional approach, one analyses the effects that a geometric symmetry of the material behavior has on the Cartesian tensor components. Typically, some of them vanishes in a particular frame, the symmetry frame, i.e. the frame whose axes coincide with the equivalent directions of the given material symmetry.

So, this approach gives a typical structure of the tensor but exclusively in the symmetry frame: the algebraic effects of this analysis are apparent only in this special frame, and vanishes in a general frame, at least apparently.

In the polar formalism, the approach is quite the opposite one: a material symmetry is intrinsically detected by a special value taken by one or more polar invariants, and this, of course, regardless of the frame in which the Cartesian components are written.

The point of view is hence clearly algebraic: the symmetry is seen as an algebraic property, and more important than its geometric description, is the effect that the invariants have on the Cartesian components and the property they represent when these invariants get the values corresponding to a symmetry.

This approach focuses hence on the algebraic effects of the symmetry and as such it is more powerful than the merely geometric one; it has allowed to discover some planar elastic symmetries unknown in the past and, studying the anisotropy of complex materials, the links that exist between the tensorial symmetries and the elastic symmetries, etc.

With the polar formalism, the classification of the elastic symmetries is strictly based upon the algebraic properties of the tensor polar invariants, not upon the geometric symmetries: then mechanical aspects assume a greater importance than the geometric ones.

This point of view lets appear a fundamental fact: to the same material symmetry, classified according to a merely geometric criterion, can belong different algebraic symmetries which have different mechanical properties.

The polar formalism apply directly to tensor components; this is why we prefer to develop the entire theory continuing to use them in place of switching immediately to the Kelvin's notation.

The transformation of Verchery

The polar formalism, as already said, is an algebraic technique based upon the use of a complex variable change.

However, unlike what done in other approaches, namely in the works of Mushkelishvili, Green & Zerna or Milne-Thomson, Verchery introduces a different transformation.

The reason is that, as we will see, this transformation allows for obtaining particularly simple matrices, namely diagonal matrices for the rotations and anti-diagonal matrices for mirror symmetries.

In short, the transformation of Verchery has better algebraic properties than the one usually introduced in the literature.

Just like Green & Zerna, Verchery introduces a complex variable change, interpreted as a change of frame:

let us consider a vector $\mathbf{x}=(x_1,x_2)$, and the transformation

$$\mathrm{X}^1=\frac{1}{\sqrt{2}}\overline{k}z,\quad \mathrm{X}^2=\overline{\mathrm{X}}^1,\quad k=e^{i\frac{\pi}{4}}, \tag{1}$$

giving the contravariant components of $\mathrm{X}^{cont}=(X^1,X^2)$, the transformed of $\mathbf{x}$ (the transformation is not orthogonal).

Equation (1) is the transformation of Verchery; z is the complex variable

$$z=x_1+ix_2. \tag{2}$$

The transformation (1) can be applied not only to rank-1 tensors, the vectors, but also to tensors of any rank.

To this purpose, it is worth to write eq. (1) in a matrix form:

$$\mathrm{X}^{cont} = \mathbf{m}_1 \mathbf{x}, \quad \rightarrow \quad \mathbf{m}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} \overline{k} & k \\ k & \overline{k} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}. \tag{3}$$

The covariant components can be easily obtained using the metric tensor $\mathbf{g}$:

$$\mathrm{X}_{cov} = \mathbf{g}_{cov}\ \mathrm{X}^{cont}, \tag{4}$$

whose components can be found expressing the length ds of an infinitesimal arc:

$$\begin{array}{l} ds^2 = dx_1^2 + dx_2^2 = dz d\overline{z} = 2\ \mathrm{dX}^1 \mathrm{dX}^2, \\ ds^2 = \mathrm{dX}^{cont} \cdot \mathbf{g}_{cov} \mathrm{dX}^{cont} = g_{ij} \mathrm{dX}^i \mathrm{dX}^j, \end{array} \rightarrow\ \mathbf{g}_{cov} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{5}$$

Hence

$$\mathbf{m}_1^{-1} = \mathbf{g}_{cov}\ \mathbf{m}_1 \tag{6}$$

and, considering eq. (1),

$$\begin{aligned} &\mathrm{X}_{cov} = (\mathrm{X}_1, \mathrm{X}_2) = (\mathrm{X}^2, \mathrm{X}^1) = (\overline{\mathrm{X}}^1, \overline{\mathrm{X}}^2) \ \rightarrow \\ &\mathrm{X}_{cov} = \overline{\mathrm{X}}^{cont} = \mathbf{m}_1^{-1}\mathbf{x}. \end{aligned} \tag{7}$$

This fact is typical of the transformation of Verchery: all the covariant components are equal to the contravariant components that are obtained swapping indexes 1 and 2, or, equivalently, they are the complex conjugate of the corresponding contravariant components, and vice-versa.

A further result for this transformation is that

$$\begin{aligned} &ds^2 = \mathrm{dX}_{cov} \cdot \mathbf{g}^{cont}\mathrm{dX}_{cov} = g^{ij}\mathrm{dX}_i\mathrm{dX}_j \ \ \rightarrow \\ &\mathbf{g}^{cont} = \mathbf{g}_{cov}^{-1} = \mathbf{g}_{cov} := \mathbf{g}. \end{aligned} \tag{8}$$

Matrix $\mathbf{m}_1$ operates the transformation of rank-1 tensors, and it has some remarkable algebraic properties, that can be readily found.

It is important to notice that these properties are shared by all the matrices $\mathbf{m}_j$ that operates the transformation for rank-j tensors.

Such properties, easy to be checked for $\mathbf{m}_1$, are:

$$
\left.\begin{aligned}
&\mathbf{m}_j^\top = \mathbf{m}_j,\\
&\overline{\mathbf{m}}_j^\top \neq \mathbf{m}_j,\\
&\mathbf{m}_j^{-1} = \overline{\mathbf{m}}_j^\top = \overline{\mathbf{m}}_j,
\end{aligned}\right. \qquad \forall j \geq 1. \tag{9}
$$

Hence, matrices $\mathbf{m}_j$ are unitary, but not Hermitian because of eq. $(9)_2$, symmetric with respect to both the diagonals and the inverse coincides with the complex conjugate[1].

[1] To make a comparison, the transformation normally used, cf. Green & Zerna, is defined by the equations $X^1 = z,\ X^2 = \overline{z}$. Following the same procedure used here for the Verchery's transformation, it is easy to check that in this case all the listed properties are no more valid.

Second-rank tensors

The matrix $\mathbf{m}_2$ for the transformation of rank-2 tensors can be computed in the following way:

$$\mathbf{m}_2 = \left[\begin{array}{c|c} m_1^{11}\mathbf{m}_1 & m_1^{12}\mathbf{m}_1 \\ \hline m_1^{21}\mathbf{m}_1 & m_1^{22}\mathbf{m}_1 \end{array}\right] = \frac{1}{2}\begin{bmatrix} -i & 1 & 1 & i \\ 1 & -i & i & 1 \\ 1 & i & -i & 1 \\ i & 1 & 1 & -i \end{bmatrix}. \tag{10}$$

It is not too hard to check that $\mathbf{m}_2$ has the properties (9).

Let us represent a second-rank tensor $\mathbf{L}$ as a column vector where the order in which the components of a tensor appear in the column is not arbitrary, but obeys to the following rule: the first component is that whose indexes are all 1 and the successive components increase the indexes starting from the right: 1111, 1112, 1121, 1122, 1211, 1212, 1221, 1222 and so on.

Then

$$
\mathsf{L}^{cont} = \mathbf{m}_2\mathbf{L} \rightarrow \left\{ \begin{array}{c} \mathsf{L}^{11} \\ \mathsf{L}^{12} \\ \mathsf{L}^{21} \\ \mathsf{L}^{22} \end{array} \right\} = \frac{1}{2} \begin{bmatrix} -i & 1 & 1 & i \\ 1 & -i & i & 1 \\ 1 & i & -i & 1 \\ i & 1 & 1 & -i \end{bmatrix} \left\{ \begin{array}{c} L_{11} \\ L_{12} \\ L_{21} \\ L_{22} \end{array} \right\}. \quad (11)
$$

As already happened for X^{cont}, we can notice that only two complex components of L^{cont} are sufficient to define $\mathbf{L}$, because

$$
\mathsf{L}^{21} = \overline{\mathsf{L}}^{12}, \quad \mathsf{L}^{22} = \overline{\mathsf{L}}^{11}. \quad (12)
$$

This is a consequence of the Verchery's transformation, valid for tensors of any rank. In addition, it is also

$$
\mathrm{tr}\ \mathbf{L} = \mathsf{L}^{12} + \mathsf{L}^{21} \quad (13)
$$

and, once put

$$
\mathbb{G} = \mathbf{g} \boxtimes \mathbf{g}, \quad (14)
$$

we get also

$$\mathsf{L}_{cov} = \mathbf{g}\mathsf{L}^{cont}\mathbf{g}^{\top} = \mathbb{G}\mathsf{L}^{cont}, \;\rightarrow\; \mathsf{L}_{ij} = g_{im}g_{jn}\mathsf{L}^{mn} \;\rightarrow$$
$$\mathsf{L}_{cov} = \begin{bmatrix} \mathsf{L}^{22} & \mathsf{L}^{21} \\ \mathsf{L}^{12} & \mathsf{L}^{11} \end{bmatrix}, \tag{15}$$

confirming what said above about the relation between covariant and contravariant components.

Remembering eqs. $(9)_3$ and (12), we then have also

$$\mathsf{L}_{cov} = \overline{\mathsf{L}}^{cont} = \begin{bmatrix} \overline{\mathsf{L}}^{11} & \overline{\mathsf{L}}^{12} \\ \overline{\mathsf{L}}^{21} & \overline{\mathsf{L}}^{22} \end{bmatrix} \;\rightarrow\; \mathsf{L}_{cov} = \mathbf{m}_2^{-1}\mathbf{L}. \tag{16}$$

In the case, interesting for us, of a symmetric second-rank tensor, eliminating the component 21, eq. (11) becomes

$$\left\{ \begin{array}{c} \mathsf{L}^{11} \\ \mathsf{L}^{12} \\ \mathsf{L}^{22} \end{array} \right\} = \frac{1}{2} \begin{bmatrix} -i & 2 & i \\ 1 & 0 & 1 \\ i & 2 & -i \end{bmatrix} \left\{ \begin{array}{c} L_{11} \\ L_{12} \\ L_{22} \end{array} \right\}. \tag{17}$$

Fourth-rank tensors

The transformation matrix $\mathbf{m}_4$ is computed as

$$\mathbf{m}_4 = \left[\begin{array}{c|c|c|c} m_2^{11}\mathbf{m}_2 & m_2^{12}\mathbf{m}_2 & m_2^{13}\mathbf{m}_2 & m_2^{14}\mathbf{m}_2 \\ \hline m_2^{21}\mathbf{m}_2 & m_2^{22}\mathbf{m}_2 & m_2^{23}\mathbf{m}_2 & m_2^{24}\mathbf{m}_2 \\ \hline m_2^{31}\mathbf{m}_2 & m_2^{32}\mathbf{m}_2 & m_2^{33}\mathbf{m}_2 & m_2^{34}\mathbf{m}_2 \\ \hline m_2^{41}\mathbf{m}_2 & m_2^{42}\mathbf{m}_2 & m_2^{43}\mathbf{m}_2 & m_2^{44}\mathbf{m}_2 \end{array}\right]. \tag{18}$$

The contravariant components of $\mathbb{T}$ can be computed as usual:

$$\mathrm{T}^{cont} = \mathbf{m}_4\mathbb{T}, \tag{19}$$

and writing $\mathbb{T}$ in the form of a column vector we get, after some rather lengthy computations,

$$
\begin{Bmatrix} \mathrm{T}^{1111} \\ \mathrm{T}^{1112} \\ \mathrm{T}^{1121} \\ \mathrm{T}^{1122} \\ \mathrm{T}^{1211} \\ \mathrm{T}^{1212} \\ \mathrm{T}^{1221} \\ \mathrm{T}^{1222} \\ \mathrm{T}^{2111} \\ \mathrm{T}^{2112} \\ \mathrm{T}^{2121} \\ \mathrm{T}^{2122} \\ \mathrm{T}^{2211} \\ \mathrm{T}^{2212} \\ \mathrm{T}^{2221} \\ \mathrm{T}^{2222} \end{Bmatrix}
=\frac{1}{4}
\begin{bmatrix}
-1 & -i & -i & 1 & -i & 1 & 1 & i & -i & 1 & 1 & i & 1 & i & i & -1 \\
-i & -1 & 1 & -i & 1 & -i & i & 1 & 1 & -i & i & 1 & i & 1 & -1 & i \\
-i & 1 & -1 & -i & 1 & i & -i & 1 & 1 & i & -i & 1 & i & -1 & 1 & i \\
1 & -i & -i & -1 & i & 1 & 1 & -i & i & 1 & 1 & -i & -1 & i & i & 1 \\
-i & 1 & 1 & i & -1 & -i & -i & 1 & 1 & i & i & -1 & -i & 1 & 1 & i \\
1 & -i & i & 1 & -i & -1 & 1 & -i & i & 1 & -1 & i & 1 & -i & i & 1 \\
1 & i & -i & 1 & -i & 1 & -1 & -i & i & -1 & 1 & i & 1 & i & -i & 1 \\
i & 1 & 1 & -i & 1 & -i & -i & -1 & -1 & i & i & 1 & i & 1 & 1 & -i \\
-i & 1 & 1 & i & 1 & i & i & -1 & -1 & -i & i & 1 & -i & 1 & 1 & i \\
1 & -i & i & 1 & i & 1 & -1 & i & -i & -1 & 1 & -i & 1 & -i & i & 1 \\
1 & i & -i & 1 & i & -1 & 1 & i & -i & 1 & -1 & -i & 1 & i & -i & 1 \\
i & 1 & 1 & -i & -1 & i & i & 1 & 1 & -i & -i & -1 & i & 1 & 1 & -i \\
1 & i & i & -1 & -i & 1 & 1 & i & -i & 1 & 1 & i & -1 & -i & -i & 1 \\
i & 1 & -1 & i & 1 & -i & i & 1 & 1 & -i & i & 1 & -i & -1 & 1 & -i \\
i & -1 & 1 & i & 1 & i & -i & 1 & 1 & i & -i & 1 & -i & 1 & -1 & -i \\
-1 & i & i & 1 & i & 1 & 1 & -i & i & 1 & 1 & -i & 1 & -i & -i & -1
\end{bmatrix}
\begin{Bmatrix} T_{1111} \\ T_{1112} \\ T_{1121} \\ T_{1122} \\ T_{1211} \\ T_{1212} \\ T_{1221} \\ T_{1222} \\ T_{2111} \\ T_{2112} \\ T_{2121} \\ T_{2122} \\ T_{2211} \\ T_{2212} \\ T_{2221} \\ T_{2222} \end{Bmatrix}. \tag{20}
$$

To check that $\mathbf{m}_4$ has the properties (9) is still rather straightforward, despite the size, 16×16, of the matrix.

Once more, only eight complex components T^{ijkl} are needed, because

$$\begin{aligned} &\mathsf{T}^{2111} = \overline{\mathsf{T}}^{1222}, \mathsf{T}^{2112} = \overline{\mathsf{T}}^{1221}, \mathsf{T}^{2121} = \overline{\mathsf{T}}^{1212}, \mathsf{T}^{2122} = \overline{\mathsf{T}}^{1211}, \\ &\mathsf{T}^{2211} = \overline{\mathsf{T}}^{1122}, \mathsf{T}^{2212} = \overline{\mathsf{T}}^{1121}, \mathsf{T}^{2221} = \overline{\mathsf{T}}^{1112}, \mathsf{T}^{2222} = \overline{\mathsf{T}}^{1111}. \end{aligned} \tag{21}$$

Also for the covariant components of $\mathbb{T}$ we get

$$\begin{aligned} &\mathsf{T}_{cov} = \mathbb{G}\mathsf{T}^{cont}\mathbb{G}^{\top} \quad \rightarrow \quad \mathsf{T}_{ijkl} = g_{im}g_{jn}g_{kp}g_{lq}\mathsf{T}^{mnpq}, \\ &\mathsf{T}_{cov} = \overline{\mathsf{T}}^{cont}, \quad \mathsf{T}_{cov} = \mathbf{m}_4^{-1}\mathbb{T} \quad \rightarrow \\ &\mathsf{T}_{1111} = \mathsf{T}^{2222} = \overline{\mathsf{T}}^{1111}, \ \mathsf{T}_{1112} = \mathsf{T}^{2221} = \overline{\mathsf{T}}^{1112}, \\ &\mathsf{T}_{1121} = \mathsf{T}^{2212} = \overline{\mathsf{T}}^{1121}, \ \text{etc.} \end{aligned} \tag{22}$$

Elasticity tensors

We consider now elasticity tensors, i.e. having the minor and major symmetries. For plane tensors, these symmetries give the following ten conditions

$$\begin{aligned} &T_{1112} = T_{1121} = T_{1211} = T_{2111}, \quad T_{1122} = T_{2211}, \\ &T_{1212} = T_{2112} = T_{2121} = T_{1221}, \quad T_{1222} = T_{2122} = T_{2212} = T_{2221}. \end{aligned} \tag{23}$$

As a consequence, there are only six independent components for a plane elastic tensor and finally we have

$$\left\{ \begin{array}{c} \mathrm{T}^{1111} \\ \mathrm{T}^{1112} \\ \mathrm{T}^{1122} \\ \mathrm{T}^{1212} \\ \mathrm{T}^{1222} \\ \mathrm{T}^{2222} \end{array} \right\} = \frac{1}{4} \begin{bmatrix} -1 & -4i & 2 & 4 & 4i & -1 \\ -i & 2 & 0 & 0 & 2 & i \\ 1 & 0 & -2 & 4 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 & 1 \\ i & 2 & 0 & 0 & 2 & -i \\ -1 & 4i & 2 & 4 & -4i & -1 \end{bmatrix} \left\{ \begin{array}{c} T_{1111} \\ T_{1112} \\ T_{1122} \\ T_{1212} \\ T_{1222} \\ T_{2222} \end{array} \right\}. \tag{24}$$

Only 4 components of T^{cont} are needed to know $\mathbb{T}$:
$\mathrm{T}^{1111}, \mathrm{T}^{1112} \in \mathbb{C}$, T^{1122} and $\mathrm{T}^{1212} \in \mathbb{R}$

Tensor rotation

We consider a new frame $\{x_1', x_2'\}$, rotated through an angle θ with respect to the initial frame $\{x_1, x_2\}$ and we pose

$$r = e^{-i\theta}, \tag{25}$$

so that in the new frame the complex variable is

$$z' = r\ z. \tag{26}$$

If we apply the Verchery's transformation (1) we get the new contravariant components of $\mathbf{x}$:

$$\begin{aligned} \mathrm{X}^{1'} &= \frac{1}{\sqrt{2}}\overline{k}\ z' = \frac{1}{\sqrt{2}}\overline{k}\ r\ z = r\ \mathrm{X}^1, \\ \mathrm{X}^{2'} &= \frac{1}{\sqrt{2}}k\ \overline{z}' = \frac{1}{\sqrt{2}}k\ \overline{r}\ \overline{z} = \overline{r}\ \mathrm{X}^2, \end{aligned} \tag{27}$$

so that we can write

$$\mathrm{X}^{cont'} = \mathbf{R}_1 \mathrm{X}^{cont} \quad \rightarrow \quad \left\{ \begin{array}{c} \mathrm{X}^{1'} \\ \mathrm{X}^{2'} \end{array} \right\} = \left[\begin{array}{cc} r & 0 \\ 0 & \overline{r} \end{array} \right] \left\{ \begin{array}{c} \mathrm{X}^{1} \\ \mathrm{X}^{2} \end{array} \right\} \tag{28}$$

The rotation matrix has a characteristic that is common to all the rotation matrices, at any tensor rank: it is diagonal.

This is a fundamental result of the Verchery's transformation because, as we will see below, it is just this property that allows for easily find tensor invariants.

The direct transformation of the real Cartesian components can be obtained using eqs. (3) and (28):

$$\mathbf{x}' = \mathbf{m}_1^{-1} \mathrm{X}^{cont\prime} = \mathbf{m}_1^{-1} \mathbf{R}_1 \mathrm{X}^{cont} = \mathbf{m}_1^{-1} \mathbf{R}_1 \mathbf{m}_1 \mathbf{x}. \tag{29}$$

Developing the calculations, one obtains

$$\mathbf{x}' = \mathbf{r}_1 \mathbf{x}, \quad \mathbf{r}_1 = \mathbf{m}_1^{-1} \mathbf{R}_1 \mathbf{m}_1 = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}; \quad c = \cos\theta, \ s = \sin\theta. \tag{30}$$

It can be noticed that $\mathbf{r}_1$ is the classical matrix for the rotation of tensors in $\mathbb{R}^2$.

The rotation matrix $\mathbf{R}_2$ for rank-two tensors can be constructed with the same rule used for $\mathbf{m}_2$, eq. (10), for finally obtaining

$$\mathsf{L}^{cont\prime} = \mathbf{R}_2 \mathsf{L}^{cont} \rightarrow \left\{ \begin{array}{c} \mathsf{L}^{11\prime} \\ \mathsf{L}^{12\prime} \\ \mathsf{L}^{21\prime} \\ \mathsf{L}^{22\prime} \end{array} \right\} = \begin{bmatrix} r^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \overline{r}^2 \end{bmatrix} \left\{ \begin{array}{c} \mathsf{L}^{11} \\ \mathsf{L}^{12} \\ \mathsf{L}^{21} \\ \mathsf{L}^{22} \end{array} \right\}. \tag{31}$$

For symmetric tensors, the above equation reduces to

$$\left\{ \begin{array}{c} \mathsf{L}^{11\prime} \\ \mathsf{L}^{12\prime} \\ \mathsf{L}^{22\prime} \end{array} \right\} = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \overline{r}^2 \end{bmatrix} \left\{ \begin{array}{c} \mathsf{L}^{11} \\ \mathsf{L}^{12} \\ \mathsf{L}^{22} \end{array} \right\}. \tag{32}$$

Also in this case, we can find the matrix $\mathbf{r}_2$ for the rotation of the real Cartesian components:

$$\mathbf{L}' = \mathbf{m}_2^{-1}\mathsf{L}^{cont'} = \mathbf{m}_2^{-1}\mathbf{R}_2\mathsf{L}^{cont} = \mathbf{m}_2^{-1}\mathbf{R}_2\mathbf{m}_2\mathbf{L} \quad \rightarrow$$

$$\mathbf{L}' = \mathbf{r}_2\mathbf{L}, \quad \mathbf{r}_2 = \mathbf{m}_2^{-1}\mathbf{R}_2\mathbf{m}_2 = \begin{bmatrix} c^2 & sc & sc & s^2 \\ -sc & c^2 & -s^2 & sc \\ -sc & -s^2 & c^2 & sc \\ s^2 & -sc & -sc & c^2 \end{bmatrix}, \tag{33}$$

which is the classical rotation matrix for rank-two tensors in the plane.

For tensors of the fourth rank, the procedure is exactly the same:

$$\mathrm{T}^{cont\prime} = \mathbf{R}_4 \mathrm{T}^{cont}, \tag{34}$$

and after some lengthy calculations we get

$$\begin{Bmatrix} \mathrm{T}^{1111\prime} \\ \mathrm{T}^{1112\prime} \\ \mathrm{T}^{1121\prime} \\ \mathrm{T}^{1122\prime} \\ \mathrm{T}^{1211\prime} \\ \mathrm{T}^{1212\prime} \\ \mathrm{T}^{1221\prime} \\ \mathrm{T}^{1222\prime} \\ \mathrm{T}^{2111\prime} \\ \mathrm{T}^{2112\prime} \\ \mathrm{T}^{2121\prime} \\ \mathrm{T}^{2122\prime} \\ \mathrm{T}^{2211\prime} \\ \mathrm{T}^{2212\prime} \\ \mathrm{T}^{2221\prime} \\ \mathrm{T}^{2222\prime} \end{Bmatrix} = \begin{bmatrix} r^4 \\ & r^2 \\ & & r^2 \\ & & & 1 \\ & & & & r^2 \\ & & & & & 1 \\ & & & & & & 1 \\ & & & & & & & \bar{r}^2 \\ & & & & & & & & r^2 \\ & & & & & & & & & 1 \\ & & & & & & & & & & 1 \\ & & & & & & & & & & & \bar{r}^2 \\ & & & & & & & & & & & & 1 \\ & & & & & & & & & & & & & \bar{r}^2 \\ & & & & & & & & & & & & & & \bar{r}^4 \\ & & & & & & & & & & & & & & & \bar{r}^2 \end{bmatrix} \begin{Bmatrix} \mathrm{T}^{1111} \\ \mathrm{T}^{1112} \\ \mathrm{T}^{1121} \\ \mathrm{T}^{1122} \\ \mathrm{T}^{1211} \\ \mathrm{T}^{1212} \\ \mathrm{T}^{1221} \\ \mathrm{T}^{1222} \\ \mathrm{T}^{2111} \\ \mathrm{T}^{2112} \\ \mathrm{T}^{2121} \\ \mathrm{T}^{2122} \\ \mathrm{T}^{2211} \\ \mathrm{T}^{2212} \\ \mathrm{T}^{2221} \\ \mathrm{T}^{2222} \end{Bmatrix}, \tag{35}$$

$$
\begin{Bmatrix} \mathsf{T}^{1111'} \\ \mathsf{T}^{1112'} \\ \mathsf{T}^{1122'} \\ \mathsf{T}^{1212'} \\ \mathsf{T}^{1222'} \\ \mathsf{T}^{2222'} \end{Bmatrix} = \begin{bmatrix} r^4 & & & & & \\ & r^2 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \overline{r}^2 & \\ & & & & & \overline{r}^4 \end{bmatrix} \begin{Bmatrix} \mathsf{T}^{1111} \\ \mathsf{T}^{1112} \\ \mathsf{T}^{1122} \\ \mathsf{T}^{1212} \\ \mathsf{T}^{1222} \\ \mathsf{T}^{2222} \end{Bmatrix}. \tag{36}
$$

Also in this case, for the rotation of the real Cartesian components we get

$$
\begin{aligned}
&\mathbb{T}' = \mathbf{m}_4^{-1}\mathsf{T}^{cont'} = \mathbf{m}_4^{-1}\mathbf{R}_4\mathsf{T}^{cont} = \mathbf{m}_4^{-1}\mathbf{R}_4\mathbf{m}_4\mathbb{T} \quad \rightarrow \\
&\mathbb{T}' = \mathbf{r}_4\mathbb{T}, \quad \mathbf{r}_4 = \mathbf{m}_4^{-1}\mathbf{R}_4\mathbf{m}_4.
\end{aligned} \tag{37}
$$

We explicit the matrix $\mathbf{r}_4$ only for the case of elasticity-like tensors:

$$\mathbf{r}_4 = \begin{bmatrix} c^4 & 4sc^3 & 2s^2c^2 & 4s^2c^2 & 4s^3c & s^4 \\ sc^3 & c^4 - 3s^2c^2 & s^3c - sc^3 & 2(s^3c - sc^3) & 3s^2c^2 - s^4 & -s^3c \\ s^2c^2 & 2(sc^3 - s^3c) & c^4 + s^4 & -4s^2c^2 & 2(s^3c - sc^3) & s^2c^2 \\ s^2c^2 & 2(sc^3 - s^3c) & -2s^2c^2 & (c^2 - s^2)^2 & 2(s^3c - sc^3) & s^2c^2 \\ s^3c & 3s^2c^2 - s^4 & sc^3 - s^3c & 2(sc^3 - s^3c) & c^4 - 3s^2c^2 & -sc^3 \\ s^4 & 4s^3c & 2s^2c^2 & 4s^2c^2 & 4sc^3 & c^4 \end{bmatrix}, \tag{38}$$

which is the classical rotation matrix for elasticity tensors in the plane.

Tensor invariants under frame rotations

To look for tensor invariants under frame rotations is particularly simple thanks to the fact that all the rotation tensors $\mathbf{R}_j$ for the contravariant complex components are diagonal, which is far to be the case for the rotation tensors $\mathbf{r}_j$ of the real Cartesian components.

This fact is the major algebraic effect of the Verchery's transformation, and motivates the method and the passage to contravariant complex components.

For better understanding the procedure, let us start with the simpler case, that of vectors; looking at eq. (28), one can see immediately that the only invariant quantity, i.e. the only quantity that can be formed using the contravariant components and whose transformation to another frame does not depend upon r, is $\mathrm{X}^1\mathrm{X}^2$.

In fact,

$$\mathsf{X}^{1\prime}\mathsf{X}^{2\prime} = r\mathsf{X}^1\, \bar{r}\mathsf{X}^2 = \mathsf{X}^1\mathsf{X}^2. \tag{39}$$

A vector has hence only a quadratic tensor invariant; using eq. (1), we get

$$\mathsf{X}^1\mathsf{X}^2 = \frac{1}{\sqrt{2}}\bar{k}z\frac{1}{\sqrt{2}}k\bar{z} = \frac{x_1^2 + x_2^2}{2}, \tag{40}$$

which is half the square of the norm of **x**, the only invariant quantity in a vector.

The same procedure can be applied to the other tensors. For **L**, eq. (31) gives two complex conjugate linear invariants, $\mathsf{L}^{12}, \mathsf{L}^{21}$, and a quadratic one, $\mathsf{L}^{11}\mathsf{L}^{22}$:

$$\begin{aligned} \mathsf{L}^{12} &= \frac{1}{2}\left[L_{11} + L_{22} - i\left(L_{12} - L_{21}\right)\right], \\ \mathsf{L}^{21} = \bar{\mathsf{L}}^{12} &= \frac{1}{2}\left[L_{11} + L_{22} + i\left(L_{12} - L_{21}\right)\right], \\ \mathsf{L}^{11}\mathsf{L}^{22} &= \frac{1}{4}\left[\left(L_{11} - L_{22}\right)^2 + \left(L_{12} + L_{21}\right)^2\right], \end{aligned} \tag{41}$$

and hence the 3 independent real invariants of **L** are

$$
\begin{aligned}
l_1 &= \mathrm{Re}\left(\mathsf{L}^{12}\right) = \mathrm{Re}\left(\mathsf{L}^{21}\right) = \frac{1}{2}\left(L_{11} + L_{22}\right) = \frac{1}{2}\mathrm{tr}\ \mathbf{L}, \\
l_2 &= \mathrm{Im}\left(\mathsf{L}^{12}\right) = \mathrm{Im}\left(\mathsf{L}^{21}\right) = \frac{1}{2}\left(L_{12} - L_{21}\right), \\
q_1 &= \frac{1}{4}\left[\left(L_{11} - L_{22}\right)^2 + \left(L_{12} + L_{21}\right)^2\right],
\end{aligned} \tag{42}
$$

which for a symmetric tensor become only two, a linear, l_1, and a quadratic one, q_1:

$$
\begin{aligned}
l_1 &= l_2 = \mathsf{L}^{12} = \mathsf{L}^{21} = \frac{1}{2}\left(L_{11} + L_{22}\right) = \frac{1}{2}\mathrm{tr}\ \mathbf{L}, \\
q_1 &= \mathsf{L}^{11}\mathsf{L}^{22} = \frac{1}{4}\left[\left(L_{11} - L_{22}\right)^2 + 4L_{12}^2\right].
\end{aligned} \tag{43}
$$

For a fourth-rank tensor $\mathbb{T}$ eq. (35) gives 43 invariants on the whole, of which 6 are linear, 17 quadratics and 20 cubics.

Nevertheless, they cannot be all independent. In fact, there can be at most 15 independent invariants for $\mathbb{T}$, because it has 16 components.

So, 28 syzygies necessarily exist among the 43 invariants.

A syzygy is a relation between two or more tensor invariants. The search for syzygies is a crucial point in determining which are the dependent invariants; unfortunately, no general method exists for finding the syzygies.

To determine all the independent invariants of a fourth-rank general tensor in $\mathbb{R}^2$ is very long and actually, it is still to be done.

For elastic tensors we have only 6 independent components, which means that there must be 5 tensor independent invariants for an elasticity tensor in $\mathbb{R}^2$. Scrutiny of eq. (122) is much simpler and it gives the following six real invariants:

$$
\begin{aligned}
L_1 &= \mathsf{T}^{1122}, \\
L_2 &= \mathsf{T}^{1212}, \\
Q_1 &= \mathsf{T}^{1111}\mathsf{T}^{2222}, \\
Q_2 &= \mathsf{T}^{1112}\mathsf{T}^{1222}, \\
C_1 + iC_2 &= \mathsf{T}^{1111}\left(\mathsf{T}^{1222}\right)^2 .
\end{aligned}
\tag{44}
$$

L_1 and L_2 are linear, Q_1 and Q_2 quadratic and C_1 and C_2 cubic.

The independent invariants are only 5$\rightarrow$ one syzygy must exist.

This is readily found observing that

$$
\begin{aligned}
C_1^2 + C_2^2 = (C_1 + iC_2)(C_1 - iC_2) = \mathsf{T}^{1111}\left(\mathsf{T}^{1222}\right)^2 \overline{\mathsf{T}}^{1111}\left(\overline{\mathsf{T}}^{1222}\right)^2 = \\
= \mathsf{T}^{1111}\left(\mathsf{T}^{1222}\right)^2 \mathsf{T}^{2222}\left(\mathsf{T}^{1112}\right)^2 = Q_1 Q_2^2.
\end{aligned}
\tag{45}
$$

In obtaining this result, we have used $\mathsf{T}^{2111} = \mathsf{T}^{1112}$ and eq. (21).

The Cartesian form of the invariants can be found by eq. (24):

$$
\begin{aligned}
L_1 &= \frac{1}{4}\left(T_{1111} - 2T_{1122} + 4T_{1212} + T_{2222}\right),\\
L_2 &= \frac{1}{4}\left(T_{1111} + 2T_{1122} + T_{2222}\right),\\
Q_1 &= \frac{1}{16}\left(T_{1111} - 2T_{1122} - 4T_{1212} + T_{2222}\right)^2 + \left(T_{1112} - T_{1222}\right)^2,\\
Q_2 &= \frac{1}{16}\left(T_{1111} - T_{2222}\right)^2 + \frac{1}{4}\left(T_{1112} + T_{1222}\right)^2,\\
C_1 &= \frac{1}{64}\left(T_{1111} - 2T_{1122} - 4T_{1212} + T_{2222}\right)\left[\left(T_{1111} - T_{2222}\right)^2 -\right.\\
&\quad \left. -4\left(T_{1112} + T_{1222}\right)^2\right] + \frac{1}{4}\left(T_{1112}^2 - T_{1222}^2\right)\left(T_{1111} - T_{2222}\right),\\
C_2 &= \frac{1}{16}\left(T_{1112} - T_{1222}\right)\left[\left(T_{1111} - T_{2222}\right)^2 - 4\left(T_{1112} + T_{1222}\right)^2\right] -\\
&\quad - \frac{1}{16}\left(T_{1112} + T_{1222}\right)\left(T_{1111} - T_{2222}\right)\left(T_{1111} - 2T_{1122} - 4T_{1212} + T_{2222}\right).
\end{aligned}
\tag{46}
$$

This result shows how it should be difficult to find the tensor invariants using the Cartesian components.

The polar components

Following the original approach of Verchery, we introduce non polynomial quantities, the polar components, better suited for anisotropic problems.

The polar components are in the same number of the independent Cartesian components, i.e. they are equal to the number of the invariants plus one: this last parameter introduces the frame orientation.

Second-rank symmetric tensors

The polar components of a symmetric second-rank tensor are introduced posing

$$\begin{aligned} \mathsf{L}^{11} &= Re^{2i\left(\Phi-\frac{\pi}{4}\right)}, \\ \mathsf{L}^{12} &= T. \end{aligned} \tag{47}$$

T and R are real quantities. They are moduli, in the sense that they are quantities having the same dimensions of the tensor they represent (e.g. the dimensions of a stress for tensor $\boldsymbol{\sigma}$).

For what concerns T, from eq. $(43)_1$ we have that

$$T = \frac{1}{2}\mathrm{tr}\ \mathbf{L} = \frac{L_{11}+L_{22}}{2}. \tag{48}$$

Being the modulus of a complex quantity, $R \geq 0$. In particular, it is

$$\mathsf{L}^{11} = Re^{2i(\Phi - \frac{\pi}{4})} = L_{12} - i\frac{L_{11} - L_{22}}{2} \;\rightarrow\; Re^{2i\Phi} = \frac{L_{11} - L_{22}}{2} + iL_{12} \tag{49}$$

and

$$R = \sqrt{\mathsf{L}^{11}\overline{\mathsf{L}}^{11}} = \sqrt{\mathsf{L}^{11}\mathsf{L}^{22}} \;\rightarrow\; R = \sqrt{\left(\frac{L_{11} - L_{22}}{2}\right)^2 + L_{12}^2} \geq 0. \tag{50}$$

$\mathsf{L}^{11}\mathsf{L}^{22}$ is an invariant, $(43)_2$; as a consequence, both T and R are invariant quantities. Φ is to be interpreted as an angle; from eq. (49),

$$\tan 2\Phi = \frac{2L_{12}}{L_{11} - L_{22}}. \tag{51}$$

Because L_{12} and $L_{11} - L_{22}$ are not invariant quantities, Φ is not an invariant, and it entirely determines the frame orientation.

Equations (48), (49) and (51) define the 3 polar components, T, R and Φ, of $\mathbf{L}$ as functions of its Cartesian components L_{ij}.

It is easy to obtain the reverse equations, that give the L_{ij}s as functions of the polar components:

$$
\begin{aligned}
L_{11} &= T + R\cos 2\Phi,\\
L_{12} &= R\sin 2\Phi,\\
L_{22} &= T - R\cos 2\Phi.
\end{aligned}
\tag{52}
$$

T represents the spherical part of $\mathbf{L}$ and R the deviatoric one, in the sense that

$$
\begin{aligned}
\mathbf{L}_{sph} &= T\mathbf{I} \quad \rightarrow \quad \|\mathbf{L}_{sph}\| = \sqrt{2}\,T,\\
\mathbf{L}_{dev} &= \mathbf{L} - \mathbf{L}_{sph} \quad \rightarrow \quad \|\mathbf{L}_{dev}\| = \sqrt{2}R.
\end{aligned}
\tag{53}
$$

Elasticity tensor

2 complex and 2 real contravariant components are sufficient to describe an elastic tensor.

Then, the polar components of a fourth-rank elasticity-type tensor are introduced putting:

$$\begin{aligned}
\mathrm{T}^{1111} &= 2R_0 e^{4i\left(\Phi_0 - \frac{\pi}{4}\right)},\\
\mathrm{T}^{1112} &= 2R_1 e^{2i\left(\Phi_1 - \frac{\pi}{4}\right)},\\
\mathrm{T}^{1122} &= 2T_0,\\
\mathrm{T}^{1212} &= 2T_1.
\end{aligned} \tag{54}$$

$T_0, T_1, R_0, R_1, \Phi_0$ and Φ_1 are the polar components of $\mathbb{T}$.

In particular, T_0, T_1, R_0 and R_1 are polar moduli, i.e. they have the dimensions of a stress, if $\mathbb{T}$ is a stiffness tensor, or the dimensions of the reciprocal of a stress, if $\mathbb{T}$ is a compliance tensor.

Moreover,

$$R_0 \geq 0, \quad R_1 \geq 0, \tag{55}$$

because they are proportional to the modulus of a complex quantity.

$\varPhi_0$ and $\varPhi_1$ are to be interpreted as polar angles; we see hence that the polar formalism gives a representation of elasticity using exclusively moduli and angles.

In this sense, it is quite different from the classical Cartesian representation, where only moduli are used, and from the representation by technical constants, which makes use of moduli and coefficients.

Using the fact that $\mathsf{T}^{2111} = \overline{\mathsf{T}}^{1222}$ etc., along with eq. (44), it is simple to show that

$$
\begin{aligned}
L_1 &= 2T_0, \\
L_2 &= 2T_1, \\
Q_1 &= 4R_0^2, \\
Q_2 &= 4R_1^2, \\
C_1 + iC_2 &= 8R_0R_1^2 e^{4i(\Phi_0-\Phi_1)} \quad \Rightarrow \\
C_1 &= 8R_0R_1^2 \cos 4(\Phi_0 - \Phi_1), \\
C_2 &= 8R_0R_1^2 \sin 4(\Phi_0 - \Phi_1).
\end{aligned}
\tag{56}
$$

This result shows that T_0, T_1, R_0, R_1 and $\Phi_0 - \Phi_1$ are tensor invariants.

They constitute a complete set of independent invariants for $\mathbb{T}$.

In particular, T_0 and T_1 are linear invariants, R_0 and R_1 are functions of quadratic invariants and $\Phi_0 - \Phi_1$ is a function of a cubic invariant, that is hence represented by a difference of angles.

The Cartesian expression of the polar components can be readily found:

$$
\begin{aligned}
8T_0 &= T_{1111} - 2T_{1122} + 4T_{1212} + T_{2222},\\
8T_1 &= T_{1111} + 2T_{1122} + T_{2222},\\
8R_0 e^{4i\Phi_0} &= T_{1111} - 2T_{1122} - 4T_{1212} + T_{2222} + 4i(T_{1112} - T_{1222}),\\
8R_1 e^{2i\Phi_1} &= T_{1111} - T_{2222} + 2i\,(T_{1112} + T_{1222})\,,
\end{aligned}
\tag{57}
$$

or, more explicitly,

$$
\begin{aligned}
T_0 &= \frac{1}{8}(T_{1111} - 2T_{1122} + 4T_{1212} + T_{2222}),\\
T_1 &= \frac{1}{8}(T_{1111} + 2T_{1122} + T_{2222}),\\
R_0 &= \frac{1}{8}\sqrt{(T_{1111} - 2T_{1122} - 4T_{1212} + T_{2222})^2 + 16(T_{1112} - T_{1222})^2},\\
R_1 &= \frac{1}{8}\sqrt{(T_{1111} - T_{2222})^2 + 4(T_{1112} + T_{1222})^2},\\
\tan 4\Phi_0 &= \frac{4(T_{1112} - T_{1222})}{T_{1111} - 2T_{1122} - 4T_{1212} + T_{2222}},\\
\tan 2\Phi_1 &= \frac{2\,(T_{1112} + T_{1222})}{T_{1111} - T_{2222}}.
\end{aligned}
\tag{58}
$$

It is apparent that the polar angles Φ_0 and Φ_1 are functions of the Cartesian components of $\mathbb{T}$ and by consequence, frame dependent, though their difference is an invariant.

Hence, the value of one of them depends upon the other one: only one of the two polar angles if free, and its choice corresponds to fix a frame.

The choice usually done is to put

$$\Phi_1 = 0, \tag{59}$$

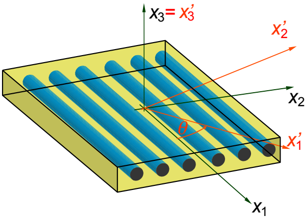
which corresponds to have the highest value of the component T_{1111} in correspondence of the axis of x_1.

Inverting eq. (57) we get:

$$
\begin{aligned}
T_{1111} &= T_0 + 2T_1 + R_0 \cos 4\Phi_0 + 4R_1 \cos 2\Phi_1,\\
T_{1112} &= R_0 \sin 4\Phi_0 + 2R_1 \sin 2\Phi_1,\\
T_{1122} &= -T_0 + 2T_1 - R_0 \cos 4\Phi_0,\\
T_{1212} &= T_0 - R_0 \cos 4\Phi_0,\\
T_{1222} &= -R_0 \sin 4\Phi_0 + 2R_1 \sin 2\Phi_1,\\
T_{2222} &= T_0 + 2T_1 + R_0 \cos 4\Phi_0 - 4R_1 \cos 2\Phi_1.
\end{aligned}
\tag{60}
$$

Change of frame

Let us consider now a change of frame from the original one $\{x_1, x_2\}$ to a frame $\{x_1', x_2'\}$ rotated counterclockwise through an angle θ, like in the figure.



Then,

$$\mathrm{L}^{11\prime} = r^2 \mathrm{L}^{11} = Re^{2i\left(\Phi - \theta - \frac{\pi}{4}\right)}, \tag{61}$$

while L^{12} does not change because it is an invariant.

So, following the usual procedure, we obtain

$$Re^{2i(\varPhi-\theta)} = \frac{L_{11}(\theta) - L_{22}(\theta)}{2} + iL_{12}(\theta), \tag{62}$$

and for the reverse equations

$$\begin{aligned} L_{11}(\theta) &= T + R\cos 2(\varPhi - \theta), \\ L_{12}(\theta) &= R\sin 2(\varPhi - \theta), \\ L_{22}(\theta) &= T - R\cos 2(\varPhi - \theta). \end{aligned} \tag{63}$$

Basically, these are just the equations of the Mohr's circle.

For an elasticity tensor, we follow the same procedure and we get

$$\begin{aligned}
\mathrm{T}^{1111'} &= r^4\mathrm{T}^{1111} = 2r^4 R_0 e^{4i\left(\Phi_0-\theta-\frac{\pi}{4}\right)},\\
\mathrm{T}^{1112'} &= r^2\mathrm{T}^{1112} = 2r^2 R_1 e^{2i\left(\Phi_1-\theta-\frac{\pi}{4}\right)},
\end{aligned} \tag{64}$$

that give

$$\begin{aligned}
8T_0 =&\ T_{1111}(\theta) - 2T_{1122}(\theta) + 4T_{1212}(\theta) + T_{2222}(\theta),\\
8T_1 =&\ T_{1111}(\theta) + 2T_{1122}(\theta) + T_{2222}(\theta),\\
8R_0 e^{4i(\Phi_0-\theta)} =&\ T_{1111}(\theta) - 2T_{1122}(\theta) - 4T_{1212}(\theta) + T_{2222}(\theta) +\\
&+ 4i\left[T_{1112}(\theta) - T_{1222}(\theta)\right],\\
8R_1 e^{2i(\Phi_1-\theta)} =&\ T_{1111}(\theta) - T_{2222}(\theta) + 2i\left[T_{1112}(\theta) + T_{1222}(\theta)\right],
\end{aligned} \tag{65}$$

and for the reverse equations

$$
\begin{aligned}
&T_{1111}(\theta)=T_0+2T_1+R_0\cos 4(\Phi_0-\theta)+4R_1\cos 2(\Phi_1-\theta),\\
&T_{1112}(\theta)=R_0\sin 4(\Phi_0-\theta)+2R_1\sin 2(\Phi_1-\theta),\\
&T_{1122}(\theta)=-T_0+2T_1-R_0\cos 4(\Phi_0-\theta),\\
&T_{1212}(\theta)=T_0-R_0\cos 4(\Phi_0-\theta),\\
&T_{1222}(\theta)=-R_0\sin 4(\Phi_0-\theta)+2R_1\sin 2(\Phi_1-\theta),\\
&T_{2222}(\theta)=T_0+2T_1+R_0\cos 4(\Phi_0-\theta)-4R_1\cos 2(\Phi_1-\theta).
\end{aligned}
\tag{66}
$$

Equations (63) and (50), when compared with Cartesian rotation matrices, show one of the greatest advantages of the polar formalism: the Cartesian components in the new frame are obtained simply subtracting the angle θ from the polar angles.

The operation of the change of frame is hence particularly simple when the Cartesian components are given as functions of the polar parameters.

Generalized Mohr's circles

It is possible to give a graphical construction corresponding to eq. (50).

This construction is called generalized Mohr's circles.

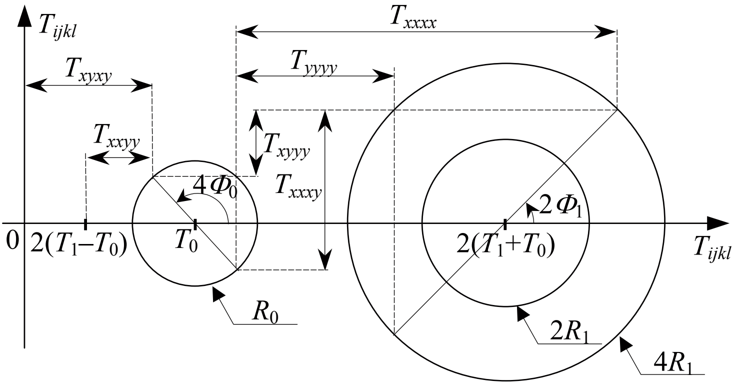


Figure: Generalized Mohr's circles.

Harmonic interpretation of the polar formalism

Let us consider, e.g., the component $T_{1111}(\theta)$

$$T_{1111}(\theta) = T_0 + 2\,T_1 + R_0 \cos 4\,(\Phi_0 - \theta) + 4 R_1 \cos 2\,(\Phi_1 - \theta) \qquad (67)$$

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- the invariants R_0 and R_1 are the factors of terms which are circular functions of 4θ and 2θ

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- the invariant $\Phi_0 - \Phi_1$ represents the phase angle between the above terms

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- the invariants R_0 and R_1 are the factors of terms which are circular functions of 4θ and 2θ
- the invariant $\Phi_0-\Phi_1$ represents the phase angle between the above terms
- R_0, R_1 and $\Phi_0-\Phi_1$ are hence the anisotropic polar invariants

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- the invariants R_0 and R_1 are the factors of terms which are circular functions of 4θ and 2θ
- the invariant $\Phi_0 - \Phi_1$ represents the phase angle between the above terms
- R_0, R_1 and $\Phi_0 - \Phi_1$ are hence the anisotropic polar invariants
- R_0 and R_1 represent, to within a factor, the amplitude of the anisotropic phases, that are directional fluctuations around the isotropic average

- The phase decomposition for all the Cartesian components

$$
\begin{aligned}
T_{1111}(\theta) &= T_0 + 2T_1 + R_0\cos 4(\Phi_0 - \theta) + 4R_1\cos 2(\Phi_1 - \theta)\\
T_{1112}(\theta) &= R_0\sin 4(\Phi_0 - \theta) + 2R_1\sin 2(\Phi_1 - \theta)\\
T_{1122}(\theta) &= -T_0 + 2T_1 - R_0\cos 4(\Phi_0 - \theta)\\
T_{1212}(\theta) &= T_0 - R_0\cos 4(\Phi_0 - \theta)\\
T_{1222}(\theta) &= -R_0\sin 4(\Phi_0 - \theta) + 2R_1\sin 2(\Phi_1 - \theta)\\
T_{2222}(\theta) &= T_0 + 2T_1 + R_0\cos 4(\Phi_0 - \theta) - 4R_1\cos 2(\Phi_1 - \theta)
\end{aligned}
$$

- The phase R_0 is the only one to be present in all the Cartesian components.

We have hence a new interpretation of anisotropic elasticity in $\mathbb{R}^2$: the anisotropic elastic behavior can be regarded as a finite sum of harmonics:

- a constant term, the isotropic phase
- the anisotropic phase, composed by two fluctuating terms:

  one varying with 2θ

  one varying with 4θ
- the amplitude of all of these phases and the phase offset of the anisotropic phases are intrinsic properties of the material, i.e. they are tensor invariants.

The above considerations give the physical meaning of the polar invariants.

We will see that these last can be linked also to two other physical facts: the elastic symmetries, determined by some special values of the polar invariants, and the strain energy decomposition.

Polar parameters of the inverse tensor

We denote the polar components of $\mathbb{S}=\mathbb{T}^{-1}$ by lower-case letters: t_0, t_1, r_0, r_1 and $\varphi_0-\varphi_1$.

These can be found expressing the Cartesian components of $\mathbb{S}$ as functions of those of $\mathbb{T}$, and these last by their polar components, eq. (50).

Comparing the result so found with eq. (50) written for $\mathbb{S}$, gives $t_0, t_1, r_0, r_1, \varphi_0$ and φ_1:

$$\begin{aligned}
t_0 &= \frac{2}{\Delta}\left(T_0T_1-R_1^2\right),\\
t_1 &= \frac{1}{2\Delta}\left(T_0^2-R_0^2\right),\\
r_0e^{4i\varphi_0} &= \frac{2}{\Delta}\left(R_1^2e^{4i\Phi_1}-T_1R_0e^{4i\Phi_0}\right),\\
r_1e^{2i\varphi_1} &= -\frac{R_1e^{2i\Phi_1}}{\Delta}\left[T_0-R_0e^{4i(\Phi_0-\Phi_1)}\right].
\end{aligned} \tag{68}$$

From the above equations, we obtain also

$$
\begin{aligned}
r_0 =& \frac{2}{\Delta}\sqrt{\left(R_1^2\cos 4\Phi_1 - T_1 R_0\cos 4\Phi_0\right)^2 + \left(R_1^2\sin 4\Phi_1 - T_1 R_0\sin 4\Phi_0\right)^2},\\
r_1 =& \frac{R_1}{\Delta}\Big\{\left[T_0\cos 2\Phi_1 - R_0\cos\left(4(\Phi_0-\Phi_1)+2\Phi_1\right)\right]^2 + \\
& \left[T_0\sin 2\Phi_1 - R_0\sin\left(4(\Phi_0-\Phi_1)+2\Phi_1\right)\right]^2\Big\}^{\frac{1}{2}},
\end{aligned}
\tag{69}
$$

and

$$
\begin{aligned}
\tan 4\varphi_0 &= \frac{R_1^2\sin 4\Phi_1 - T_1 R_0\sin 4\Phi_0}{R_1^2\cos 4\Phi_1 - T_1 R_0\cos 4\Phi_0},\\
\tan 2\varphi_1 &= \frac{T_0\sin 2\Phi_1 - R_0\sin\left[4(\Phi_0-\Phi_1)+2\Phi_1\right]}{T_0\cos 2\Phi_1 - R_0\cos\left[4(\Phi_0-\Phi_1)+2\Phi_1\right]}.
\end{aligned}
\tag{70}
$$

$\varDelta$ is an invariant quantity, defined by

$$\begin{aligned}\varDelta &= 8\,T_1\left(T_0^2-R_0^2\right)-16R_1^2\left[T_0-R_0\cos 4\left(\varPhi_0-\varPhi_1\right)\right]=\\&=\det\begin{bmatrix}T_{1111} & T_{1122} & T_{1112}\\ & T_{2222} & T_{1222}\\ sym & & T_{1212}\end{bmatrix}.\end{aligned} \tag{71}$$

We will see that $\varDelta$ is a positive quantity.

We can switch $\mathbb{T}$ and $\mathbb{S}$ $\rightarrow$

$$R_1=0\Leftrightarrow r_1=0,\qquad R_0=0\not\Leftrightarrow r_0=0. \tag{72}$$

This has a considerable importance in the determination of all the elastic symmetries

Technical constants and polar invariants

We can now express the technical constants as functions of the polar invariants.

First, we write $\mathbb{S}$ in terms of the compliance polar invariants:

$$\begin{aligned}
S_{1111}(\theta)&=t_0+2t_1+r_0\cos 4(\varphi_0-\theta)+4r_1\cos 2(\varphi_1-\theta),\\
S_{1112}(\theta)&=r_0\sin 4(\varphi_0-\theta)+2r_1\sin 2(\varphi_1-\theta),\\
S_{1122}(\theta)&=-t_0+2t_1-r_0\cos 4(\varphi_0-\theta),\\
S_{1212}(\theta)&=t_0-r_0\cos 4(\varphi_0-\theta),\\
S_{1222}(\theta)&=-r_0\sin 4(\varphi_0-\theta)+2r_1\sin 2(\varphi_1-\theta),\\
S_{2222}(\theta)&=t_0+2t_1+r_0\cos 4(\varphi_0-\theta)-4r_1\cos 2(\varphi_1-\theta).
\end{aligned} \tag{73}$$

Now we inject the above expressions for the S_{ijkl} in the definitions of the technical constants:

- Young's moduli:

$$
\begin{aligned}
E_1(\theta) &= \frac{1}{S_{1111}(\theta)} = \frac{1}{t_0+2t_1+r_0\cos 4\left(\varphi_0-\theta\right)+4r_1\cos 2\left(\varphi_1-\theta\right)};\\
E_2(\theta) &= \frac{1}{S_{1111}(\theta)} = \frac{1}{t_0+2t_1+r_0\cos 4\left(\varphi_0-\theta\right)-4r_1\cos 2\left(\varphi_1-\theta\right)};
\end{aligned}
\tag{74}
$$

- shear modulus:

$$
G_{12}(\theta) = \frac{1}{4S_{1212}(\theta)} = \frac{1}{4[t_0-r_0\cos 4\left(\varphi_0-\theta\right)]}; \tag{75}
$$

- Poisson's coefficient:

$$
\nu_{12}(\theta) = -\frac{S_{1122}(\theta)}{S_{1111}(\theta)} = \frac{t_0-2t_1+r_0\cos 4\left(\varphi_0-\theta\right)}{t_0+2t_1+r_0\cos 4\left(\varphi_0-\theta\right)+4r_1\cos 2\left(\varphi_1-\theta\right)}; \tag{76}
$$

- coefficients of mutual influence of the first type:

$$\eta_{1,12}(\theta) = \frac{S_{1112}(\theta)}{2S_{1212}(\theta)} = \frac{r_0 \sin 4(\varphi_0 - \theta) + 2r_1 \sin 2(\varphi_1 - \theta)}{2\left[t_0 - r_0 \cos 4(\varphi_0 - \theta)\right]},$$

$$\eta_{2,12}(\theta) = \frac{S_{1222}(\theta)}{2S_{1212}(\theta)} = \frac{-r_0 \sin 4(\varphi_0 - \theta) + 2r_1 \sin 2(\varphi_1 - \theta)}{2\left[t_0 - r_0 \cos 4(\varphi_0 - \theta)\right]}; \tag{77}$$

- coefficients of mutual influence of the second type:

$$\eta_{12,1}(\theta) = 2\frac{S_{1112}(\theta)}{S_{1111}(\theta)} = 2\frac{r_0 \sin 4(\varphi_0 - \theta) + 2r_1 \sin 2(\varphi_1 - \theta)}{t_0 + 2t_1 + r_0 \cos 4(\varphi_0 - \theta) + 4r_1 \cos 2(\varphi_1 - \theta)},$$

$$\eta_{12,2}(\theta) = 2\frac{S_{1222}(\theta)}{2S_{2222}(\theta)} = 2\frac{-r_0 \sin 4(\varphi_0 - \theta) + 2r_1 \sin 2(\varphi_1 - \theta)}{t_0 + 2t_1 + r_0 \cos 4(\varphi_0 - \theta) - 4r_1 \cos 2(\varphi_1 - \theta)}. \tag{78}$$

Using eq. (68) it is also possible to express the technical constants as functions of the stiffness polar invariants; in the most general case, this leads to very long expressions, that we omit here.

Nevertheless, it is interesting to consider the case of isotropic materials; for such a situation, eq. (68) reduce to

$$t_0 = \frac{1}{4\,T_0}, \quad t_1 = \frac{1}{16\,T_1}, \quad r_0 = 0, \quad r_1 = 0, \tag{79}$$

so we get

- Young's modulus:

$$E = \frac{1}{t_0 + 2t_1} = \frac{8\,T_0\,T_1}{T_0 + 2\,T_1}; \tag{80}$$

- shear modulus:

$$G = \frac{1}{4t_0} = T_0; \tag{81}$$

- Poisson's coefficient:

$$\nu = \frac{t_0 - 2t_1}{t_0 + 2t_1} = \frac{2T_1 - T_0}{2T_1 + T_0}. \tag{82}$$

The remaining coefficients are of course null for isotropic materials. Another modulus is usually introduced for isotropic materials: the bulk modulus κ:

$$\forall\ \boldsymbol{\sigma} = p\mathbf{I}, \quad \kappa := \frac{p}{\mathrm{tr}\boldsymbol{\varepsilon}}. \tag{83}$$

Applying this definition to the plane anisotropic case gives

$$\kappa = \frac{1}{S_{1111}(\theta) + 2S_{1122}(\theta) + S_{2222}(\theta)} = \frac{1}{8t_1}, \tag{84}$$

which, for a material at least square symmetric ($R_1 = r_1 = 0$), gives also

$$\kappa = 2T_1 \tag{85}$$

We have hence a physical meaning for the polar invariants of isotropy:

- t_0 and T_0 are linked to the shear modulus
- t_1 and T_1 are related to the bulk modulus

We will see that the existence of these 2 different parts of the isotropic phase corresponds to the physical fact that for classical elastic materials the whole of the strain energy can be split, under some conditions, into two different parts, a spherical and a deviatoric one, the first linked to volume changes, and ruled by the bulk modulus, hence by T_1, the other by the shear modulus, hence by T_0 (for the isotropic case).

The relations between the Lamé's constants and the polar invariants can also be given:

$$\kappa = \lambda + \mu,\ G = \mu \ \Rightarrow\ \lambda = 2T_1 - T_0,\ \mu = T_0 \tag{86}$$

Polar decomposition of the strain energy

Let us consider a layer subjected to some stresses $\boldsymbol{\sigma}$, whose polar components are T, R and Φ,that produce the strain ε, described by its polar components t, r and φ.

Then the strain energy V is

$$V = \frac{1}{2}\boldsymbol{\sigma} \cdot \varepsilon = T\ t + R\ r \cos 2(\Phi - \varphi). \tag{87}$$

Using the polar formalism for ε and $\boldsymbol{\sigma}$ we get easily

$$\begin{aligned} V_s &:= \frac{1}{2}\varepsilon_{sph} \cdot \boldsymbol{\sigma}_{sph} = T\ t, \\ V_d &:= \frac{1}{2}\varepsilon_{dev} \cdot \boldsymbol{\sigma}_{dev} = R\ r \cos 2(\Phi - \varphi). \end{aligned} \tag{88}$$

We introduce now the material behavior, using T_0, T_1, R_0, R_1 and $\Phi_0 - \Phi_1$ for representing $\mathbb{E}$):

$$
\begin{aligned}
V = \frac{1}{2}\boldsymbol{\varepsilon}\cdot\mathbb{E}\boldsymbol{\varepsilon} = 4\,T_1 t^2 + 8R_1\cos 2(\varPhi_1-\varphi)r\ \ t+ \\
+\,2\left[T_0 + R_0\cos 4(\varPhi_0-\varphi)\right] r^2.
\end{aligned} \tag{89}
$$

The variation δV caused by a variation $\delta\boldsymbol{\varepsilon}$ of the deformation is

$$
\delta V = \boldsymbol{\sigma}\cdot\delta\boldsymbol{\varepsilon} = 2\,T\delta t + 2R\cos 2(\varPhi-\varphi)\delta r + 4R\ r\sin 2(\varPhi-\varphi)\delta\varphi, \tag{90}
$$

and hence the spherical and deviatoric parts of $\boldsymbol{\sigma}$ are

$$
\begin{aligned}
T &= \frac{1}{2}\frac{\partial V}{\partial t},\\
Re^{2i\varPhi} &= \frac{1}{2}\left(\frac{\partial V}{\partial r} + \frac{i}{2r}\frac{\partial V}{\partial\varphi}\right)e^{2i\varphi}.
\end{aligned} \tag{91}
$$

Injecting eq. (90) in eq. (91) gives

$$\begin{aligned} T &= 4T_1 t + 4R_1 r\cos 2(\Phi_1-\varphi),\\ Re^{2i\Phi} &= 2T_0 re^{2i\varphi} + 2R_0 re^{2i(2\Phi_0-\varphi)} + 4R_1 te^{2i\Phi_1}. \end{aligned} \tag{92}$$

The above relations show a fact previously discussed: for an anisotropic material, also in $\mathbb{R}^2$, in the most general case the spherical and deviatoric parts of $\boldsymbol{\sigma}$ depend on both the spherical and deviatoric parts of $\boldsymbol{\varepsilon}$.

Using these relations in the expressions of V_s and V_d gives

$$\begin{aligned} V_s &= 4T_1 t^2 + 4R_1 r\ t\cos 2(\Phi_1-\varphi),\\ V_d &= 2r^2\left[T_0 + R_0\cos 4(\Phi_0-\varphi)\right] + 4R_1 r\ t\cos 2(\Phi_1-\varphi). \end{aligned} \tag{93}$$

We can then observe the role played by the different polar invariants of $\mathbb{E}$ in the decomposition of the strain energy: T_1 affects only V_s, T_0 and R_0 only V_d while R_1 couples V_s with V_d.

For materials with $R_1 = 0$, the two parts are uncoupled. It is then clear, and simple to be checked, that when $R_1 = 0$ [2]

$$\boldsymbol{\sigma}_{dev} = \mathbb{E}\boldsymbol{\varepsilon}_{dev}, \quad \boldsymbol{\sigma}_{sph} = \mathbb{E}\boldsymbol{\varepsilon}_{sph} \ \Rightarrow \tag{94}$$

$$\begin{aligned} V_s = V_{sph} \quad &\rightarrow \quad \frac{1}{2}\boldsymbol{\varepsilon}_{sph}\cdot\boldsymbol{\sigma}_{sph} = \frac{1}{2}\boldsymbol{\varepsilon}_{sph}\cdot\mathbb{E}\boldsymbol{\varepsilon}_{sph},\\ V_d = V_{dev} \quad &\rightarrow \quad \frac{1}{2}\boldsymbol{\varepsilon}_{dev}\cdot\boldsymbol{\sigma}_{dev} = \frac{1}{2}\boldsymbol{\varepsilon}_{dev}\cdot\mathbb{E}\boldsymbol{\varepsilon}_{dev}, \end{aligned} \tag{95}$$

which implies

$$V = V_{sph} + V_{dev} = V_s + V_d. \tag{96}$$

Finally, the minimal requirement, in $\mathbb{R}^2$, for decomposing the strain energy in a spherical and deviatoric part is $R_1 = 0$, confirming a general result already found in 3D elasticity for the cubic syngony.

[2]

$$V_{sph} := \frac{1}{2}\boldsymbol{\varepsilon}_{sph}\cdot\mathbb{E}\boldsymbol{\varepsilon}_{sph}, \quad V_{dev} := \frac{1}{2}\boldsymbol{\varepsilon}_{dev}\cdot\mathbb{E}\boldsymbol{\varepsilon}_{dev}.$$

Bounds on the polar invariants

The positiveness of the strain energy V gives the bounds on the components of $\mathbb{E}$, so also on its polar invariants.

V is a quadratic form of r and t, eq.(89), that can be written as

$$V = \{r, t\} \cdot \begin{bmatrix} 2\left[T_0 + R_0 \cos 4(\Phi_0 - \varphi)\right] & 4R_1 \cos 2(\Phi_1 - \varphi) \\ 4R_1 \cos 2(\Phi_1 - \varphi) & 4T_1 \end{bmatrix} \begin{Bmatrix} r \\ t \end{Bmatrix}. \tag{97}$$

$V > 0 \ \forall \{r, t\}$ if and only if the matrix in the previous equation is positive definite.

This happens[3] $\iff$,

$$\begin{aligned} &T_0 + R_0 \cos 4(\Phi_0 - \varphi) > 0, \\ &T_1\left[T_0 + R_0 \cos 4(\Phi_0 - \varphi)\right] > 2R_1^2 \cos^2 2(\Phi_1 - \varphi), \end{aligned} \qquad \forall \varphi. \tag{98}$$

[3] See Theorem on the leading principal minors

To be noticed that, because the term at the second member of eq. $(98)_2$ is a square, hence a nonnegative quantity, if eq. $(98)_1$ is satisfied then it is also

$$T_1 > 0. \tag{99}$$

We can obtain relations on the only polar invariants as follows: first, we transform eq. $(98)_2$ introducing the angle

$$\alpha = \Phi_1 - \varphi \tag{100}$$

which implies

$$\Phi_0 - \varphi = \Delta\Phi + \alpha, \tag{101}$$

where

$$\Delta\Phi = \Phi_0 - \Phi_1. \tag{102}$$

Equation $(98)_2$ becomes hence

$$T_1 \left[T_0 + R_0 \cos 4(\Delta\Phi + \alpha) \right] > 2R_1^2 \cos^2 2\alpha \quad \forall \alpha, \tag{103}$$

that can be transformed, using standard trigonometric identities, first to

$$T_0 T_1 - R_1^2 + \left\{\left[T_1 R_0 \cos 4\varDelta\varPhi - R_1^2\right]\cos 4\alpha - T_1 R_0 \sin 4\varDelta\varPhi \sin 4\alpha\right\} > 0 \quad \forall\alpha, \tag{104}$$

then to

$$T_0 T_1 - R_1^2 > \sqrt{(T_1 R_0 \cos 4\varDelta\varPhi - R_1^2)^2 + T_1^2 R_0^2 \sin^2 4\varDelta\varPhi} \cos 4(\alpha - \varpi) \quad \forall\alpha, \tag{105}$$

where

$$\varpi = \frac{1}{4}\arctan\frac{T_1 R_0 \sin 4\varDelta\varPhi}{R_1^2 - T_1 R_0 \cos 4\varDelta\varPhi}, \tag{106}$$

a function of only invariants of $\mathbb{E}$.

The quantity under the square root in (105) is strictly positive$\Rightarrow$ eqs. $(98)_1$ and (105) to be true $\forall\varphi$ resume, with some simple manipulations, to

$$\begin{aligned} &T_0 - R_0 > 0,\\ &T_0 T_1 - R_1^2 > 0,\\ &T_1(T_0^2 - R_0^2) - 2R_1^2\left[T_0 - R_0 \cos 4\varDelta\varPhi\right] > 0. \end{aligned} \tag{107}$$

Condition $(107)_2$ is less restrictive than condition $(107)_3$, and can be discarded. To show this, let us transform eq. (107) to a dimensionless form upon introduction of the ratios

$$\xi = \frac{T_0 T_1}{R_1^2}, \ \eta = \frac{R_0}{T_0}. \tag{108}$$

To remark that by eqs. (55), (99) and $(107)_1$ and because $r \geq 0$, ξ and η cannot be negative quantities. Introducing eq. (108) into eq. (107) gives

$$\eta < 1, \quad \xi > 1, \quad \xi > 2\frac{1 - \eta \cos 4\Delta\Phi}{1 - \eta^2}. \tag{109}$$

Then, condition $(109)_3$ is more restrictive than condition $(109)_2$ if

$$2\frac{1 - \eta \cos 4\Delta\Phi}{1 - \eta^2} \geq 1, \tag{110}$$

thanks to (109_1) equivalent to

$$\eta^2 - 2\eta \cos 4\Delta\Phi + 1 \geq 0, \tag{111}$$

which is always true, as it is easily checked.

Finally, condition $(107)_2$ can be discarded because less restrictive than condition $(107)_3$ and the only invariant conditions for positive definiteness of $\mathbb{E}$ are eqs. $(107)_{1,3}$, along with the two conditions (55), intrinsic to the polar method:

$$
\begin{aligned}
& T_0 - R_0 > 0, \\
& T_1(T_0^2 - R_0^2) - 2R_1^2\left[T_0 - R_0 \cos 4(\Phi_0 - \Phi_1)\right] > 0, \\
& R_0 \geq 0, \\
& R_1 \geq 0.
\end{aligned}
\tag{112}
$$

To remark also that conditions (112) imply that the isotropic part of $\mathbb{E}$ is strictly positive:

$$
T_0 > 0, \quad T_1 > 0. \tag{113}
$$

The above four intrinsic conditions (112) are valid for a completely anisotropic planar material.

Finally, we notice that eq. $(112)_2$ is equivalent to state that Δ, eq. (71), is necessarily a positive quantity.

Symmetries

We ponder now the way the elastic symmetries for tensor $\mathbb{T}$ can be described within the polar formalism.

Quantities L_1, L_2, Q_1, Q_2, C_1 and C_2 are tensor invariants under the action of a frame rotation.

Nevertheless, a symmetry with respect to an axis inclined of the angle α on the axis of x_1 does not leave unchanged all of these quantities.

This can be seen in the following way: such a symmetry is described by the complex variable transformation

$$z'' = s^2\overline{z}, \qquad s = e^{i\alpha}; \tag{114}$$

applying the Verchery's transformation we get

$$\begin{aligned} \mathrm{X}^{1''} &= \frac{1}{\sqrt{2}}\overline{k}\ z'' = \frac{1}{\sqrt{2}}\overline{k}\ s^2\overline{z} = -i\ s^2\mathrm{X}^2, \\ \mathrm{X}^{2''} &= \frac{1}{\sqrt{2}}k\ \overline{z}'' = \frac{1}{\sqrt{2}}k\ \overline{s}^2\ z = i\ \overline{s}^2\mathrm{X}^1. \end{aligned} \tag{115}$$

In matrix form

$$\mathrm{X}^{cont''} = \mathbf{S}_1 \mathrm{X}^{cont} \;\rightarrow\; \left\{ \begin{array}{c} \mathrm{X}^{1''} \\ \mathrm{X}^{2''} \end{array} \right\} = \left[\begin{array}{cc} 0 & -i\ s^2 \\ i\ \bar{s}^2 & 0 \end{array} \right] \left\{ \begin{array}{c} \mathrm{X}^{1} \\ \mathrm{X}^{2} \end{array} \right\}. \tag{116}$$

This result shows that $\mathrm{X}^1\mathrm{X}^2$ is still the only invariant for a vector: a mirror symmetry does not affect the norm of a vector.

The symmetry matrix has a typical structure, given by the Verchery's transformation: it is anti-diagonal.

This is true for the symmetry matrices of any rank tensors, that can be constructed using the same procedure of matrices $\mathbf{m}_j$.

We obtain hence, for rank-two tensors

$$\mathsf{L}^{cont\prime\prime} = \mathbf{S}_2 \mathsf{L}^{cont} \rightarrow \begin{Bmatrix} \mathsf{L}^{11\prime\prime} \\ \mathsf{L}^{12\prime\prime} \\ \mathsf{L}^{21\prime\prime} \\ \mathsf{L}^{22\prime\prime} \end{Bmatrix} = \begin{bmatrix} & & & -s^4 \\ & & 1 & \\ & 1 & & \\ -\overline{s}^4 & & & \end{bmatrix} \begin{Bmatrix} \mathsf{L}^{11} \\ \mathsf{L}^{12} \\ \mathsf{L}^{21} \\ \mathsf{L}^{22} \end{Bmatrix}, \tag{117}$$

which shows that a symmetry does not add any more information: $\mathsf{L}^{12}, \mathsf{L}^{21}$ and $\mathsf{L}^{11}\mathsf{L}^{22}$ are still tensor invariants also under a mirror symmetry.

In other words, mirror symmetries have no effects on plane rank-two tensors.

Fourth rank tensors:

$$\mathrm{T}^{cont''} = \mathbf{S}_4 \mathrm{T}^{cont} \;\rightarrow$$

$$\begin{Bmatrix} \mathrm{T}^{1111''} \\ \mathrm{T}^{1112''} \\ \mathrm{T}^{1121''} \\ \mathrm{T}^{1122''} \\ \mathrm{T}^{1211''} \\ \mathrm{T}^{1212''} \\ \mathrm{T}^{1221''} \\ \mathrm{T}^{1222''} \\ \mathrm{T}^{2111''} \\ \mathrm{T}^{2112''} \\ \mathrm{T}^{2121''} \\ \mathrm{T}^{2122''} \\ \mathrm{T}^{2211''} \\ \mathrm{T}^{2212''} \\ \mathrm{T}^{2221''} \\ \mathrm{T}^{2222''} \end{Bmatrix} = \begin{bmatrix}
 & & & & & & & & & & & & & & & s^8 \\
 & & & & & & & & & & & & & & -s^4 & \\
 & & & & & & & & & & & & & -s^4 & & \\
 & & & & & & & & & & & & 1 & & & \\
 & & & & & & & & & & & -s^4 & & & & \\
 & & & & & & & & & & 1 & & & & & \\
 & & & & & & & & & 1 & & & & & & \\
 & & & & & & & & -\bar{s}^4 & & & & & & & \\
 & & & & & & & -s^4 & & & & & & & & \\
 & & & & & & 1 & & & & & & & & & \\
 & & & & & 1 & & & & & & & & & & \\
 & & & & -\bar{s}^4 & & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & & & \\
 & & -\bar{s}^4 & & & & & & & & & & & & & \\
 & -\bar{s}^4 & & & & & & & & & & & & & & \\
\bar{s}^8 & & & & & & & & & & & & & & &
\end{bmatrix} \begin{Bmatrix} \mathrm{T}^{1111} \\ \mathrm{T}^{1112} \\ \mathrm{T}^{1121} \\ \mathrm{T}^{1122} \\ \mathrm{T}^{1211} \\ \mathrm{T}^{1212} \\ \mathrm{T}^{1221} \\ \mathrm{T}^{1222} \\ \mathrm{T}^{2111} \\ \mathrm{T}^{2112} \\ \mathrm{T}^{2121} \\ \mathrm{T}^{2122} \\ \mathrm{T}^{2211} \\ \mathrm{T}^{2212} \\ \mathrm{T}^{2221} \\ \mathrm{T}^{2222} \end{Bmatrix}, \tag{118}$$

that for an elasticity tensor becomes

$$\mathrm{T}^{cont''} = \mathbf{S}_4 \mathrm{T}^{cont} \quad \rightarrow$$

$$\left\{ \begin{array}{c} \mathrm{T}^{1111''} \\ \mathrm{T}^{1112''} \\ \mathrm{T}^{1122''} \\ \mathrm{T}^{1212''} \\ \mathrm{T}^{1222''} \\ \mathrm{T}^{2222''} \end{array} \right\} = \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & s^8 \\ 0 & 0 & 0 & 0 & -s^4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\overline{s}^4 & 0 & 0 & 0 & 0 \\ \overline{s}^8 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \left\{ \begin{array}{c} \mathrm{T}^{1111} \\ \mathrm{T}^{1112} \\ \mathrm{T}^{1122} \\ \mathrm{T}^{1212} \\ \mathrm{T}^{1222} \\ \mathrm{T}^{2222} \end{array} \right\}, \quad (119)$$

where the anti-diagonal structure is only apparently lost, due to the removed components.

A scrutiny of eq. (119) shows immediately that L_1, L_2, Q_1 and Q_2 are still invariants also under the action of a mirror symmetry.

This is not the case for C_1 and C_2:

$$\begin{aligned} C_1'' + iC_2'' = \mathsf{T}^{1111''}\left(\mathsf{T}^{1222''}\right)^2 = s^8\mathsf{T}^{2222}\left(\bar{s}^4\mathsf{T}^{1112}\right)^2 = \\ = \overline{\mathsf{T}}^{1111}\left(\overline{\mathsf{T}}^{1222}\right)^2 = C_1 - iC_2 : \end{aligned} \tag{120}$$

C_2 is antisymmetric as effect of the mirror symmetry.

To study the effect of the mirror symmetry, we operate a rotation of axes, choosing the new frame so that the bisector of the first quadrant coincide with the axes of mirror symmetry.

For such a choice, it must be

$$\theta = \alpha - \frac{\pi}{4} \quad \Rightarrow \quad r = k\,\bar{s}. \tag{121}$$

Then,

$$\begin{Bmatrix} \mathsf{T}^{1111'} \\ \mathsf{T}^{1112'} \\ \mathsf{T}^{1122'} \\ \mathsf{T}^{1212'} \\ \mathsf{T}^{1222'} \\ \mathsf{T}^{2222'} \end{Bmatrix} = \begin{bmatrix} -e^{-4i\alpha} & & & & & \\ & ie^{-2i\alpha} & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & -ie^{2i\alpha} & \\ & & & & & -e^{4i\alpha} \end{bmatrix} \begin{Bmatrix} \mathsf{T}^{1111} \\ \mathsf{T}^{1112} \\ \mathsf{T}^{1122} \\ \mathsf{T}^{1212} \\ \mathsf{T}^{1222} \\ \mathsf{T}^{2222} \end{Bmatrix} \tag{122}$$

For the same choice of the new frame, the axes of x_1' and x_2' are equivalent with respect to the mirror symmetry, which implies

$$\begin{aligned} \mathsf{T}^{1111'} = \mathsf{T}^{2222'} = \overline{\mathsf{T}}^{1111'}, \\ \mathsf{T}^{1112'} = \mathsf{T}^{1222'} = \overline{\mathsf{T}}^{1112'}, \end{aligned} \tag{123}$$

and hence that

$$
\begin{aligned}
\mathrm{T}^{1111'} &= -e^{-4i\alpha}\mathrm{T}^{1111} \in \mathbb{R} \\
\mathrm{T}^{1222'} &= -ie^{2i\alpha}\mathrm{T}^{1222} \in \mathbb{R};
\end{aligned} \tag{124}
$$

by consequence, for the cubic invariants we get

$$
C_1 + iC_2 = \mathrm{T}^{1111}\left(\mathrm{T}^{1222}\right)^2 = \mathrm{T}^{1111'}\left(\mathrm{T}^{1222'}\right)^2 \in \mathbb{R} \quad \Rightarrow \quad C_2 = 0. \tag{125}
$$

This result opens the way to examine the algebraic characterization of elastic symmetries in $\mathbb{R}^2$.

First of all, we remark that if α is the direction of an axis of symmetry, then $\beta = \alpha + \pi/2$ is also the direction of an axis of symmetry.

In fact, if the direction of β becomes the bisector of a new frame $\{x_1'', x_2''\}$, then $x_1'' = x_2', x_2'' = -x_1'$: the axes x_1'' and x_2'' are, of course, still equivalent with respect to a mirror symmetry, that can be only that of β, their bisector.

This fact just shows that in $\mathbb{R}^2$ the monoclinic syngony cannot exist, the minimal symmetry condition being that of the orthorhombic syngony, i.e. of orthotropic tensors $\mathbb{T}$.

The direction of the mirror can be obtained considering that the imaginary part of $\mathsf{T}^{1222'}$ must be null:

$$
\begin{gathered}
\mathrm{Im}\left(\mathsf{T}^{1222'}\right) = \mathrm{Im}\left(-ie^{2i\alpha}\mathsf{T}^{1222}\right) = 0 \quad \Rightarrow \\
\tan 2\alpha = \frac{\mathrm{Re}\left(\mathsf{T}^{1222}\right)}{\mathrm{Im}\left(\mathsf{T}^{1222}\right)} = \frac{2(T_{1112} + T_{1222})}{T_{1111} - T_{2222}}.
\end{gathered} \tag{126}
$$

The general condition for the existence of a mirror symmetry and hence, for what said above, for the tensor $\mathbb{T}$ to be orthotropic, is eq. (125): $C_2 = 0$. The syzygy becomes then

$$C_1^2 = Q_1 Q_2^2 \quad \Rightarrow \quad Q_1 = \left(\frac{C_1}{Q_2}\right)^2. \tag{127}$$

so that in case of orthotropy, there are only four independent nonzero invariants: L_1, L_2, Q_2 and C_1[4].

The above equation let us obtain the general algebraic relation characterizing all the types of elastic symmetry in $\mathbb{R}^2$:

$$R_0 R_1^2 \sin 4(\Phi_0 - \Phi_1) = 0 \tag{128}$$

[4]It is important to preserve, in the set of the independent invariants, the invariant of the highest degree, that is why we keep C_1 in the list.

Such condition depends upon three invariants, $R_0, R_1, \Phi_0 - \Phi_1$, and can be satisfied when these invariants take some special values.

To each value of one of the above three invariants root of eq. (128) corresponds a different case of elastic symmetry in $\mathbb{R}^2$.

To remark that condition (128) is an intrinsic characterization of elastic symmetries in $\mathbb{R}^2$, because it makes use of only tensor invariants.

So, all the following special cases are also intrinsic conditions of orthotropy and so on.

Let us consider all of them separately.

Ordinary orthotropy

The first solution to (128) that we consider is

$$\sin 4(\Phi_0 - \Phi_1) = 0 \;\Rightarrow\; \Phi_0 - \Phi_1 = K\frac{\pi}{4},\; K \in \{0,1\} \;\Rightarrow\; C_2 = 0 \;\Rightarrow$$
$$(T_{1112} - T_{1222})\left[(T_{1111} - T_{2222})^2 - 4\,(T_{1112} + T_{1222})^2\right] -$$
$$(T_{1112} + T_{1222})(T_{1111} - T_{2222})(T_{1111} - 2T_{1122} - 4T_{1212} + T_{2222}) = 0. \tag{129}$$

Condition (129) depends upon a cubic invariant[5].

It characterizes intrinsically ordinary orthotropy as the particular anisotropic situation where the shift angle between the two anisotropy phases is a multiple of $\pi/4$; due to the periodicity of the functions, only 2 cases are meaningful: 0 or $\pi/4$.

[5] This is the first invariant characterization of orthotropy in $\mathbb{R}^2$ and was explicitly given by Verchery & Vong in 1986

This result shows that, generally speaking, for the same set of invariants T_0, T_1, R_0 and R_1 two possible and distinct orthotropic materials can exist: one with $K=0$ and the other one with $K=1$.

This fact is interesting *per se* and because it shows that an algebraic analysis of symmetries, based upon the study of the invariants, gives more information than a mere geometric study.

If a frame rotation of Φ_1 is operated (which corresponds to chose the frame where $\Phi_1=0$), eq. (50) can be written as

$$\begin{aligned}
T_{1111}(\theta)&=T_0+2T_1+(-1)^K R_0\cos 4\theta+4R_1\cos 2\theta,\\
T_{1112}(\theta)&=-(-1)^K R_0\sin 4\theta-2R_1\sin 2\theta,\\
T_{1122}(\theta)&=-T_0+2T_1-(-1)^K R_0\cos 4\theta,\\
T_{1212}(\theta)&=T_0-(-1)^K R_0\cos 4\theta,\\
T_{1222}(\theta)&=(-1)^K R_0\sin 4\theta-2R_1\sin 2\theta,\\
T_{2222}(\theta)&=T_0+2T_1+(-1)^K R_0\cos 4\theta-4R_1\cos 2\theta.
\end{aligned}\tag{130}$$

The parameter K, that is an invariant, characterizes ordinary orthotropy; its importance has been observed in different studies.

In particular K plays a fundamental role in several optimization problems: an optimal solution to a given problem becomes the anti-optimal, i.e. the worst one, when K switches from 0 to 1 and vice-versa.

To have an idea of the influence of parameter K, i.e. of the type of ordinary orthotropy, let us consider two examples.

Example 1: variation of the normal stiffness, i.e. of the component $T_{1111}(\theta)$, eq. $(130)_1$. We want to know of which type is its variation with θ: how much are its stationary points, where they are located etc.

The derivatives of $T_{1111}(\theta)$ are

$$\begin{aligned} \frac{dT_{1111}}{d\theta} &= -8R_1\left[(-1)^K\rho\cos 2\theta+1\right]\sin 2\theta, \\ \frac{d^2T_{1111}}{d\theta^2} &= -16R_1\left[(-1)^K\rho\cos 4\theta+\cos 2\theta\right], \end{aligned} \tag{131}$$

where

$$\rho=\frac{R_0}{R_1} \tag{132}$$

is a dimensionless parameter called the anisotropy ratio which measures the relative importance of the two anisotropy phases.

From eq. $(131)_1$ we find that possible stationary points are

$$\theta_1=0,\quad \theta_2=\frac{1}{2}\arccos\frac{(-1)^{K+1}}{\rho},\quad \theta_3=\frac{\pi}{2}, \tag{133}$$

with the solution θ_2 that exists if and only if $\rho>1$. For these roots,

$$
\begin{aligned}
T_{1111}(\theta_1) &= T_0 + 2T_1 + (-1)^K R_0 + 4R_1, \\
T_{1111}(\theta_2) &= T_0 + 2T_1 - (-1)^K \left(R_0 + 2\frac{R_1}{\rho} \right), \\
T_{1111}(\theta_3) &= T_0 + 2T_1 + (-1)^K R_0 - 4R_1,
\end{aligned} \tag{134}
$$

We remark also that for $K = 0,\ \theta_2 \in [\pi/4, \pi/2)$, while for $K = 1,\ \theta_2 \in (0, \pi/4[$. Also,

$$
\begin{aligned}
\left.\frac{d^2 T_{1111}}{d\theta^2}\right|_{\theta_1} &= -16R_1 \left[(-1)^K \rho + 1 \right], \\
\left.\frac{d^2 T_{1111}}{d\theta^2}\right|_{\theta_2} &= -16R_1 (-1)^K \frac{1-\rho^2}{\rho}, \\
\left.\frac{d^2 T_{1111}}{d\theta^2}\right|_{\theta_3} &= -16R_1 \left[(-1)^K \rho - 1 \right].
\end{aligned} \tag{135}
$$

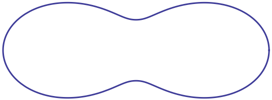
The results are summarized in the following Table.

It can be remarked that the intermediary stationary point changes from a global minimum to a global maximum when K changes from 0 to 1.

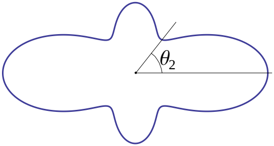
Table: Stationary points of $T_{1111}(\theta)$ for ordinary orthotropy in $\mathbb{R}^2$.

$K=0$			
$\rho \leq 1$	θ_1	Global max:	$T_{1111} = T_0 + 2T_1 + R_0 + 4R_1$
	θ_3	Global min:	$T_{1111} = T_0 + 2T_1 + R_0 - 4R_1$
$\rho > 1$	θ_1	Global max:	$T_{1111} = T_0 + 2T_1 + R_0 + 4R_1$
	θ_2	Global min:	$T_{1111} = T_0 + 2T_1 - R_0 - 2\frac{R_1}{\rho}$
	θ_3	Local max:	$T_{1111} = T_0 + 2T_1 + R_0 - 4R_1$
$K=1$			
$\rho \leq 1$	θ_1	Global max:	$T_{1111} = T_0 + 2T_1 - R_0 + 4R_1$
	θ_3	Global min:	$T_{1111} = T_0 + 2T_1 - R_0 - 4R_1$
$\rho > 1$	θ_1	Local max:	$T_{1111} = T_0 + 2T_1 - R_0 + 4R_1$
	θ_2	Global max:	$T_{1111} = T_0 + 2T_1 + R_0 + 2\frac{R_1}{\rho}$
	θ_3	Global min:	$T_{1111} = T_0 + 2T_1 - R_0 - 4R_1$

$\rho < 1, K = 0, 1$



$\rho > 1, K = 0$



$\rho > 1, K = 1$

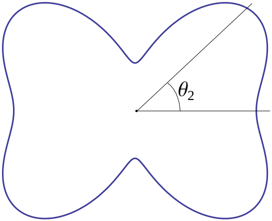


Figure: Different cases of $T_{1111}(\theta)$ for ordinary orthotropy in $\mathbb{R}^2$.

Example 2: a plate is formed by bonding together two identical orthotropic layers. The problem is to find the orientation angles $\delta_1 \neq \delta_2$ of the two layers that maximize the shear stiffness G_{12}.

G_{12} is simply the average of the moduli T_{1212} of the two layers, to be written in the same common frame:

$$G_{12} = \frac{1}{2}\left[T_{1212}(\delta_1) + T_{1212}(\delta_2)\right], \tag{136}$$

that with the polar formalism becomes

$$G_{12} = T_0 - (-1)^K R_0 \eta, \qquad \eta = \frac{\cos 4\delta_1 + \cos 4\delta_2}{2}, \quad -1 \leq \eta \leq 1. \tag{137}$$

G_{12}^{max} is get for $\eta = -1$ if $K = 0$, but for $\eta = 1$ if $K = 1$.

In both the cases, $G_{12}^{max} = T_0 + R_0$.

Because it must be $\delta_1 \neq \delta_2$, the solution for the case $K = 0$ is $\delta_1 = \pm\pi/4, \delta_2 = -\delta_1$, while for the case $K = 1$ it is $\delta_1 = 0, \delta_2 = \pi/2$ (or indifferently $\delta_1 = \pi/2, \delta_2 = 0$).

It can be also remarked what already said about the effect of K: in both the cases, the optimal solution for a value of K is the anti-optimal one for the other K: $G_{12}^{min} = T_0 - R_0$, obtained for $\eta = 1$ when $K = 0$ and for $\eta = -1$ when $K = 1$.

The two cases of $K = 0$ or $K = 1$ corresponds to what Pedersen names high ($K = 1$) or low ($K = 0$) shear modulus materials.

The above example shows the reason of such a denomination, but the former example as well as the results of other studies on K, reveal that its importance is far greater than that of a mere distinction of orthotropic layers based upon the value of their shear modulus.

Two questions concern $\mathbb{S}$, the inverse of $\mathbb{T}$: how is it oriented the orthotropy of $\mathbb{S}$ and of which type is it?

To this purpose, the inverse equations giving r_0 and r_1 after a rotation of Φ_1 become

$$\begin{aligned} r_0 e^{4i(\varphi_0-\Phi_1)} &= \frac{2}{\Delta}\left[R_1^2 - T_1 R_0 e^{4i(\Phi_0-\Phi_1)}\right], \\ r_1 e^{2i(\varphi_1-\Phi_1)} &= -\frac{R_1}{\Delta}\left[T_0 - R_0 e^{4i(\Phi_0-\Phi_1)}\right], \end{aligned} \tag{138}$$

and, because $\mathbb{T}$ is orthotropic, eq. (129),

$$\begin{aligned} r_0 e^{4i(\varphi_0-\Phi_1)} &= \frac{2}{\Delta}\left[R_1^2 - (-1)^K T_1 R_0\right], \\ r_1 e^{2i(\varphi_1-\Phi_1)} &= -\frac{1}{\Delta}R_1\left[T_0 - (-1)^K R_0\right]. \end{aligned} \tag{139}$$

Both the right-hand terms in eq. (139) $\in \mathbb{R} \Rightarrow$

$$\begin{aligned} \sin 4(\varphi_0 - \Phi_1) = 0 \quad &\Rightarrow \quad \varphi_0 = \Phi_1 + \beta_0 \frac{\pi}{4}, \\ \sin 2(\varphi_1 - \Phi_1) = 0 \quad &\Rightarrow \quad \varphi_1 = \Phi_1 + \beta_1 \frac{\pi}{2}, \end{aligned} \qquad \beta_0, \beta_1 \in \{0, 1\}. \tag{140}$$

Let us consider first φ_1: the real part of eq. $(139)_2$ is

$$r_1 \cos 2(\varphi_1 - \Phi_1) = (-1)^{\beta_1} r_1 = -\frac{1}{\Delta} R_1 \left[T_0 - (-1)^K R_0 \right]. \tag{141}$$

In the above equation, it is

$$T_0 - (-1)^K R_0 > 0, \quad \Delta > 0, \quad R_1 > 0, \quad r_1 > 0, \tag{142}$$

then, it is necessarily

$$\beta_1 = 1 \quad \Rightarrow \quad \varphi_1 = \Phi_1 + \frac{\pi}{2}. \tag{143}$$

This result states that $\mathbb{S}$ is always turned of $\pi/2$ with respect to $\mathbb{T}$.

We pass now to analyze φ_0: the real part of eq. $(139)_1$ is

$$r_0 \cos 4(\varphi_0 - \Phi_1) = \frac{2}{\Delta}\left[R_1^2 - (-1)^K T_1 R_0\right] \quad \Rightarrow$$
$$(-1)^{\beta_0} = \frac{2}{r_0 \Delta}\left[R_1^2 - (-1)^K T_1 R_0\right]. \tag{144}$$

Both the quantities Δ and r_0 are positive, so:

$$\beta_0 = 0 \iff R_1^2 - (-1)^K T_1 R_0 > 0 \rightarrow \begin{cases} K = 0: & R_1^2 - T_1 R_0 > 0, \\ K = 1: & R_1^2 + T_1 R_0 > 0 \text{ always.} \end{cases} \tag{145}$$

By consequence

$$\beta_0 = 0 \;\Rightarrow\; \varphi_0 = \Phi_1 \;\text{ when }\; \begin{cases} K = 0 \;\text{ and }\; R_1^2 > T_1 R_0, \\ \text{or} \\ K = 1, \end{cases}$$
$$\beta_0 = 1 \;\Rightarrow\; \varphi_0 = \Phi_1 + \frac{\pi}{4} \;\text{ when }\; K = 0 \;\text{ and }\; R_1^2 < T_1 R_0. \tag{146}$$

Then, the difference between the two polar angles of $\mathbb{S}$ can be only

$$\varphi_0 - \varphi_1 = (\beta_0 - 2)\frac{\pi}{4}, \tag{147}$$

$\Rightarrow$ $\mathbb{T}$ is ordinarily orthotropic $\iff$ $\mathbb{S}$ is.

Hence, putting, as already done for $\mathbb{T}$,

$$\varphi_0 - \varphi_1 = k\frac{\pi}{4}, \quad k = \beta_0 - 2, \tag{148}$$

we get that

$$\left.\begin{array}{l} K = 0 \ \text{ and } \ R_1^2 > T_1 R_0 \\ \text{or} \\ K = 1 \end{array}\right\} \Rightarrow \quad k = 0, \tag{149}$$
$$K = 0 \ \text{ and } \ R_1^2 < T_1 R_0 \quad \Rightarrow \quad k = 1.$$

Finally, an elasticity tensor and its inverse, when ordinarily orthotropic, can be of a different type; in particular, the possible combinations are three: $(K = 0, k = 0)$, $(K = 0, k = 1)$, $(K = 1, k = 0)$.

The bounds on polar invariants in the case of ordinarily orthotropic materials become

$$\begin{aligned} &T_0 > R_0, \\ &T_1 \left[T_0 + (-1)^K R_0 \right] > 2R_1^2, \\ &R_0 \geq 0, \\ &R_1 \geq 0. \end{aligned} \tag{150}$$

Equation $(150)_2$ suggests a graphical representation: the level lines of the surface

$$\mathcal{S} = \frac{2R_1^2}{T_1} \tag{151}$$

are the intersection with the planes

$$T_0 + (-1)^K R_0 = \gamma. \tag{152}$$

For the same T_0 and R_0, the constant γ takes the values

$$\gamma_0 = T_0 + R_0 \ \text{ for } \ K = 0, \quad \gamma_1 = T_0 - R_0 \ \text{ for } \ K = 1, \tag{153}$$

with of course $\gamma_0 > \gamma_1$.

So, the two planes intersect the surface $\mathcal{S}$ through two different level curves, the one corresponding to $K = 0$ higher than that of $K = 1$, see the figure.

As a consequence, if for a couple T_1, R_1 condition $(150)_2$ is satisfied for $K = 0$, it is possible that the same is not true when $K = 1$.

In this sense, materials with $K = 1$ are less probable than materials with $K = 0$, nonetheless they can exist.

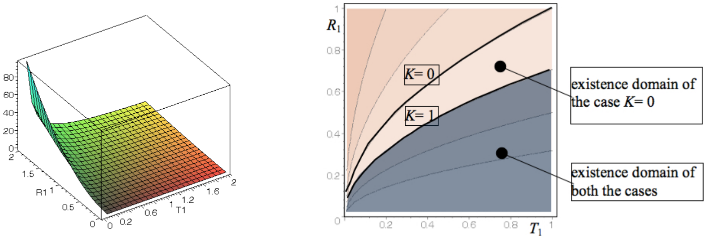


Figure: Existence domains of the two types of ordinary orthotropy in $\mathbb{R}^2$.

Finally, we have seen that what is commonly considered the ordinary orthotropy in $\mathbb{R}^2$ is actually composed by two distinct cases, that have quite different mechanical properties.

This type of symmetry is identified by a cubic invariant, that in the end can be represented by a simple integer, K, which can get only two values, 0 and 1.

It is possible that for a same material, the stiffness and the compliance tensors are ordinarily orthotropic of different types.

Special orthotropies

The general equation of elastic symmetries in $\mathbb{R}^2$

$$R_0 R_1^2 \sin 4(\Phi_0 - \Phi_1) = 0 \tag{154}$$

can be satisfied also by other conditions than root (129).

Algebraically speaking, unlike in the case of ordinary orthotropy, detected by a cubic invariant, all the other solutions are linked to special values get by quadratic invariants and they are characterized by the vanishing of at least one of the two anisotropic phases.

For these reasons, such cases of elastic symmetry are called special orthotropies, besides the last case, that of isotropy.

R_0-orthotropy

A root of eq. (154) is

$$R_0 = 0 \tag{155}$$

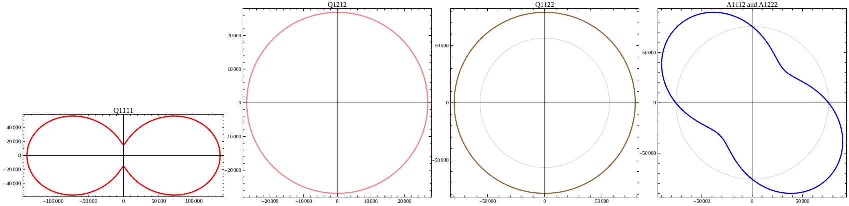
This equation identifies a special orthotropy, the so-called *$R_0 -$ orthotropy* (PV, J of Elas, 2002).

The discovery of this type of special orthotropy has been done thanks to the polar formalism and it constitutes a rather strange case of elastic behavior, whose existence has been later discovered also in $\mathbb{R}^3$ (R. Forte, 2005).

It is easily recognized that

$$R_0 = 0 \ \ \Rightarrow \ \ \begin{cases} Q_1 = C_1 = 0, \ \ \mathrm{T}^{1111} = \mathrm{T}^{2222} = 0, \\ (T_{1111} - 2T_{1122} - 4T_{1212} + T_{2222})^2 + 16(T_{1112} - T_{1222})^2 = 0. \end{cases} \tag{156}$$

Like ordinary orthotropy, this case presents 2 orthogonal axes of symmetry, but it has some peculiar characteristics:



The Cartesian components are (we fix the frame putting $\Phi_1 = 0$)

$$\begin{aligned}
T_{1111}(\theta) &= T_0 + 2T_1 + 4R_1 \cos 2\theta,\\
T_{1112}(\theta) &= -\,2R_1 \sin 2\theta,\\
T_{1122}(\theta) &= -T_0 + 2T_1,\\
T_{1212}(\theta) &= T_0,\\
T_{1222}(\theta) &= -\,2R_1 \sin 2\theta,\\
T_{2222}(\theta) &= T_0 + 2T_1 - 4R_1 \cos 2\theta.
\end{aligned} \tag{157}$$

- the anisotropic phase depending on R_0 is absent $\Rightarrow$
  - T_{1122} and T_{1212}, are isotropic
  - the other components depend upon the circular functions of $2\theta \rightarrow$ they change like the components of a 2^{nd}-rank tensor
- unlike what happens in all the other cases of anisotropy, $T_{1112}(\theta) = T_{1222}(\theta)\ \forall\theta$
- only 3 invariants are nonzero: L_1, L_2 and Q_2
- the polar angle Φ_0 is now meaningless
- this case of orthotropy is not characterized by a special value of the phase angle between the two anisotropic phases, but by the absence of one of them

Let us now consider what happens for the compliance tensor $\mathbb{S} = \mathbb{T}^{-1}$: when $R_0 = 0$, eq. (68) becomes

$$
\begin{aligned}
t_0 &= \frac{T_0 T_1 - R_1^2}{4T_0(T_0 T_1 - 2R_1^2)},\\
t_1 &= \frac{T_0}{16(T_0 T_1 - 2R_1^2)},\\
r_0 e^{4i\varphi_0} &= \frac{R_1^2 e^{4i\Phi_1}}{4T_0(T_0 T_1 - 2R_1^2)},\\
r_1 e^{2i\varphi_1} &= -\frac{R_1 e^{2i\Phi_1}}{8(T_0 T_1 - 2R_1^2)},
\end{aligned}
\tag{158}
$$

By consequence

$$
\begin{aligned}
r_0 &= \frac{R_1}{4T_0(T_0T_1 - 2R_1^2)}, \qquad && \varphi_0 = \Phi_1, \\
r_1 &= \frac{R_1}{8(T_0T_1 - 2R_1^2)}, && \varphi_1 = \Phi_1 + \frac{\pi}{2}.
\end{aligned}
\tag{159}
$$

As already remarked $R_0 = 0 \not\Rightarrow r0 = 0$: $\mathbb{S}$ depends on both the anisotropic phases, that is, its components preserve a higher degree of symmetry than those of $\mathbb{T}$.

This is a rather unusual case, where stiffness and compliance of the same material do not have the same kind of variation, the same morphology.

In addition, tensor $\mathbb{S}$ has always $k = 0$.

Nevertheless, just like $\mathbb{T}$, also $\mathbb{S}$ depends upon only 3 independent nonzero invariants, because

$$r_0 = \frac{r_1^2}{t_1}. \tag{160}$$

Hence, once a frame chosen fixing Φ_1, φ_0 and φ_1 are fixed too, and the only polar moduli t_0, t_1 and r_1 are sufficient to completely determine $\mathbb{S}$. If $\Phi_1 = 0$,

$$\begin{aligned}
S_{1111} &= t_0 + 2t_1 + \frac{r_1^2}{t_1}\cos 4\theta - 4r_1 \cos 2\theta,\\
S_{1112} &= -\frac{r_1^2}{t_1}\sin 4\theta + 2r_1 \sin 2\theta,\\
S_{1122} &= -t_0 + 2t_1 - \frac{r_1^2}{t_1}\cos 4\theta,\\
S_{1212} &= t_0 - \frac{r_1^2}{t_1}\cos 4\theta,\\
S_{1222} &= \frac{r_1^2}{t_1}\sin 4\theta + 2r_1 \sin 2\theta,\\
S_{2222} &= t_0 + 2t_1 + \frac{r_1^2}{t_1}\cos 4\theta + 4r_1 \cos 2\theta,
\end{aligned} \tag{161}$$

or, injecting eq. (158) into the previous equation,

$$
\begin{aligned}
S_{1111} &= \frac{1}{8(T_0T_1-2R_1^2)}\left[T_0+2T_1+2\frac{R_1^2}{T_0}(\cos 4\theta-1)-4R_1\cos 2\theta\right],\\
S_{1112} &= \frac{R_1}{4(T_0T_1-2R_1^2)}\left(-\frac{R_1}{T_0}\sin 4\theta+\sin 2\theta\right),\\
S_{1122} &= \frac{1}{8(T_0T_1-2R_1^2)}\left[T_0-2T_1-2\frac{R_1^2}{T_0}(\cos 4\theta-1)\right],\\
S_{1212} &= \frac{1}{4(T_0T_1-2R_1^2)}\left[T_1-\frac{R_1^2}{T_0}(\cos 4\theta+1)\right],\\
S_{1222} &= \frac{R_1}{4(T_0T_1-2R_1^2)}\left(\frac{R_1}{T_0}\sin 4\theta+\sin 2\theta\right),\\
S_{2222} &= \frac{1}{8(T_0T_1-2R_1^2)}\left[T_0+2T_1+2\frac{R_1^2}{T_0}(\cos 4\theta-1)+4R_1\cos 2\theta\right].
\end{aligned}
\tag{162}
$$

Contrarily to what happens for $\mathbb{T}$, S_{1122} and S_{1212} are not isotropic and $S_{1112} \neq S_{1222}$; nevertheless, just as for any common orthotropic layer, both them are null in correspondence of the two symmetry axes.

The general bounds (112) become, for R_0-orthotropy,

$$T_0 > \frac{2R_1^2}{T_1}, \quad R_1 > 0, \tag{163}$$

hence only 2 intrinsic bounds are sufficient.

Finally, one can wonder if R_0-orthotropic materials do really exist.

Actually, they do; in fact, it is rather simple, using the polar formalism and the classical lamination theory, to see that a R_0-orthotropic lamina can be fabricated reinforcing an isotropic matrix by unidirectional fibers arranged in equal quantity along two directions tilted of 45.

A special property of R_0-orthotropic layers, is linked to the sensitivity of a laminate to layers' orientation defects.

It has been shown (PV, J of Elas, 2001) that the influence of such defects on the uncoupling and quasi-homogeneity of a laminate[6] depends on the anisotropy ratio ρ, eq. (132).

In particular, the sensitivity to uncoupling or quasi-homogeneity is minimal when $\rho = 0$, i.e. when the laminate is composed by R_0-orthotropic layers.

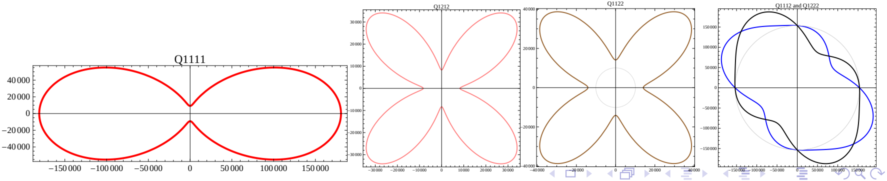
[6] A laminate is said to be quasi-homogeneous if the bending and extension response are uncoupled and equal (PV, IJSS 2001)

r_0-orthotropy

It has already been noticed that relations (68) are perfectly symmetric, i.e., they can be rewritten swapping the polar compliance constants with the polar stiffness constants, i.e., putting upper-case letters at the left-hand side and lower-case letters at the right-hand side of relations (68).

This circumstance, together with the fact that whenever $R_0 = 0$, then $r_0 \neq 0$, implies the existence of another special orthotropy, an analog of R_0-orthotropy, but concerning compliance, not stiffness: it will be indicated in the following as r_0-orthotropy (PV, J of Elas, 2002).

So, we can see that a R_0-orthotropic layer is not also r_0-orthotropic, and vice-versa.



In this sense, special orthotropies of the type R_0 are more a symmetry of a tensor than that of a material, in the sense that a material, e.g., R_0-orthotropic, has a compliance tensor that, at least apparently[7], has a common orthotropic behavior: the orthotropy axes do not change from stiffness to compliance, but the mechanical behavior is different in the two cases.

Of course, all the remarks done and results found in the previous section for R_0-orthotropy are still valid for r_0-orthotropy, with the exception of the study of $E_1(\theta)$, because the reciprocal of T_{1111} is meaningless, it is sufficient to change the lower-case letters with capital letters to all the polar components and the word stiffness with the word compliance.

[7] Apparently because if one makes experimental tests on the components of $\mathbb{S}$ or traces the directional diagrams of its components, they look like those of an ordinarily orthotropic material with $k=0$, the difference is in the special value get by r_0, eq. (160).

Something different can be said about the technical constants; in fact, putting $\varphi_1 = 0$, the compliance tensor $\mathbb{S}$ looks like

$$\begin{aligned}
S_{1111}(\theta) &= t_0 + 2t_1 + 4r_1 \cos 2\theta,\\
S_{1112}(\theta) &= -2r_1 \sin 2\theta,\\
S_{1122}(\theta) &= -t_0 + 2t_1,\\
S_{1212}(\theta) &= t_0,\\
S_{1222}(\theta) &= -2r_1 \sin 2\theta,\\
S_{2222}(\theta) &= t_0 + 2t_1 - 4r_1 \cos 2\theta,
\end{aligned} \tag{164}$$

which gives

$$\begin{aligned}
E_1(\theta) &= \frac{1}{S_{1111}(\theta)} = \frac{1}{t_0 + 2t_1 + 4r_1 \cos 2\theta},\\
G_{12}(\theta) &= \frac{1}{4S_{1212}(\theta)} = \frac{1}{4t_0},\\
\nu_{12}(\theta) &= -\frac{S_{1122}(\theta)}{S_{1111}(\theta)} = \frac{t_0 - 2t_1}{t_0 + 2t_1 + 4r_1 \cos 2\theta},\\
\eta_{1,12}(\theta) &= \eta_{2,12}(\theta) = \frac{S_{1222}(\theta)}{S_{1212}(\theta)} = -2\frac{r_1 \sin 2\theta}{t_0}.
\end{aligned} \tag{165}$$

We can hence remark that $E_1(\theta)$, $\nu_{12}(\theta)$ and $\eta_{1,12}(\theta)$ vary with 2θ, while the shear modulus $G_{12}(\theta)$ is isotropic.

This is a basic characteristic of r_0-orthotropic materials.

It was observed experimentally since the fifties that paper has this characteristic. Only recently an explanation of this fact in the framework of classical elasticity has been done, thanks to the polar formalism (PV, J of Elas, 2010).

Just like for R_0-orthotropy, only 3 nonzero independent invariants are sufficient to completely determine $\mathbb{S}$:t_0, t_1, r_1.

From eq. (165) we get also

$$
\begin{aligned}
t_0 &= \frac{1}{4G_{12}},\\
t_1 &= \frac{1}{2}\left(\frac{1}{4G_{12}} - \frac{\nu_{12}}{E_1}\right),\\
r_1 &= \frac{1}{4}\left(\frac{1+\nu_{12}}{E_1} - \frac{1}{2G_{12}}\right).
\end{aligned}
\tag{166}
$$

The general bounds (112) become, for r_0-orthotropy,

$$t_0 > \frac{2r_1^2}{t_1}, \quad r_1 > 0 \tag{167}$$

like in the case of R_0-orthotropy, so also in this case, of course, only 2 intrinsic bounds are sufficient.

Using eq. (166), the above bounds can be rewritten also in terms of technical constants:

$$\begin{aligned} &\frac{1+\nu_{12}}{E_1} - \frac{1}{2G_{12}} > 0, \\ &E_1 > G_{12}(1+\nu_{12})^2. \end{aligned} \tag{168}$$

Finally, just like for the previous case of R_0-orthotropic materials, it is easy to see that for the stiffness tensor it is

$$R_0 = \frac{R_1^2}{T_1}, \quad K = 0. \tag{169}$$

Square symmetry

Another root of eq. (154), is

$$R_1 = 0. \tag{170}$$

Just like the case of R_0-orthotropy, also in this case an anisotropy phase, the one varying with 2θ, vanishes, so it is a special orthotropy, determined once more by a quadratic invariant:

$$R_1 = 0 \quad \Rightarrow \quad \begin{cases} Q_2 = C_1 = 0, \ \mathrm{T}^{1112} = \mathrm{T}^{1222} = 0, \\ (T_{1111} - T_{2222})^2 + 4(T_{1112} + T_{1222})^2 = 0. \end{cases} \tag{171}$$

The only nonzero invariants are L_1, L_2 and Q_1.

In this case, the polar angle Φ_1 is meaningless, so the frame can be fixed only fixing a value for Φ_0.

Choosing $\Phi_0 = 0$, the Cartesian components of $\mathbb{T}$ are

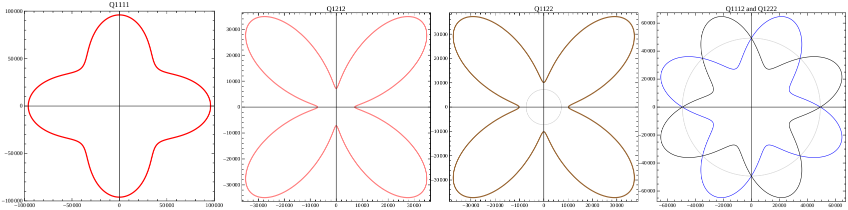
$$
\begin{aligned}
&T_{1111}(\theta) = T_0 + 2T_1 + R_0 \cos 4\theta,\\
&T_{1112}(\theta) = -R_0 \sin 4\theta,\\
&T_{1122}(\theta) = -T_0 + 2T_1 - R_0 \cos 4\theta,\\
&T_{1212}(\theta) = T_0 - R_0 \cos 4\theta,\\
&T_{1222}(\theta) = R_0 \sin 4\theta,\\
&T_{2222}(\theta) = T_0 + 2T_1 + R_0 \cos 4\theta.
\end{aligned}
\tag{172}
$$

We can remark that all the components are periodic of $\pi/2$:

$$
T_{ijkl}\left(\theta + \frac{\pi}{2}\right) = T_{ijkl}(\theta) \quad \forall \theta. \tag{173}
$$

For this reason, this special orthotropy is known in the literature as square symmetry and actually, it is the corresponding, in $\mathbb{R}^2$, of the cubic syngony.

This fact can be immediately appreciated looking at the directional diagram of its components, see the figure.



These materials can be fabricated reinforcing an isotropic matrix with a balanced fabric, i.e. by a fabric having the same amount of fibers in warp and weft.

We remark also that components T_{1122} and T_{1212} are the same of the case of ordinary orthotropy with $K = 0$ and that

$$T_{1111}(\theta) = T_{2222}(\theta), \qquad T_{1112}(\theta) = -T_{1222}(\theta) \quad \forall\theta. \tag{174}$$

Because everything is periodic of $\pi/2$, there is another couple of mirror symmetry axes, tilted of $\pi/4$ with respect to the directions $\Phi_0, \Phi_0 + \pi/2$.

In fact, eq. (124), the direction α of the mirror symmetry is given by

$$\begin{gathered}\mathrm{Im}\left(\mathrm{T}^{1111'}\right) = \mathrm{Im}\left(-ie^{2i\alpha}\mathrm{T}^{1111}\right) = 0 \quad \Rightarrow \\ \tan 4\alpha = \tan 4\left(\alpha + \frac{\pi}{4}\right) = \frac{\mathrm{Re}\left(\mathrm{T}^{1111}\right)}{\mathrm{Im}\left(\mathrm{T}^{1111}\right)} = \frac{T_{1111} - 2T_{1122} - 4T_{1212} + T_{2222}}{4(T_{1112} - T_{1222})}.\end{gathered} \tag{175}$$

Unlike the case of R_0-orthotropy, when a material has $R_1 = 0$ it has also $r_1 = 0$: square symmetry is a property of both the stiffness and the compliance tensors.

Also, for square symmetric materials, tensors $\mathbb{T}$ and $\mathbb{S}$ preserve the typical variation with the orientation: their components vary with 4θ.

The general bound for the polar invariants (112) become now

$$\begin{aligned} &T_1(T_0 - R_0) > 0, \\ &R_0 \geq 0. \end{aligned} \tag{176}$$

Isotropy

The last possible syngony for a planar material is isotropy; in this case, every angle α must determine the direction of a mirror symmetry.

This means that α must be, at the same time, the solution of eq. (126) and of eq. (175), which gives the condition

$$\begin{aligned}&\mathrm{T}^{1111}=\mathrm{T}^{1112}=0 \quad\Rightarrow\quad Q_1=Q_2=C_1=0, \quad\Rightarrow\quad R_0=R_1=0 \quad\Rightarrow\\ &T_{1112}=T_{1222}=0, \quad T_{2222}=T_{1111}, \quad T_{1111}=T_{1122}+2T_{1212}.\end{aligned} \tag{177}$$

Algebraically, isotropy is hence characterized by the fact that the two anisotropy phases vanish

It can be remarked also that a material is isotropic if and only if the conditions for the two special orthotropies are satisfied at the same time: algebraically, isotropy is determined by the vanishing of two quadratic invariants.

Alternatively, isotropy can be determined by a unique condition in place of the two polar relations $R_0 = R_1 = 0$,

$$
\begin{aligned}
& R_0^2 + R_1^2 = 0 \;\Rightarrow \\
& \left[(T_{1111} - 2T_{1122} - 4T_{1212} + T_{2222})^2 + 16(T_{1112} - T_{1222})^2\right]^2 + \\
& \left[(T_{1111} - T_{2222})^2 + 4(T_{1112} + T_{1222})^2\right]^2 = 0
\end{aligned}
\tag{178}
$$

which makes use of a fourth degree invariant.

Some general remarks on elastic symmetries in $\mathbb{R}^2$

The results found in the previous Sections, deserve some commentary:

- from a purely geometric point of view, i.e. merely considering the elastic symmetries, nothing differentiate ordinary orthotropy from the special orthotropy $R_0 = 0$: both of them have only a couple of mutually orthogonal symmetry axes.
- From the algebraic point of view, they are different: they depend upon a different number of independent nonzero invariants and they are determined by invariant conditions concerning invariants of a different order.
- They also are interpreted differently: ordinary orthotropy corresponds to a precise value taken by the phase angle between the two anisotropic phases, R_0-orthotropy to the vanishing of the anisotropic phase varying with 4θ.

- Also, while ordinary orthotropy preserves the same morphology also for the inverse tensor, though it is possible a change of type, from $K = 0$ to $k = 1$, R_0-orthotropy does not preserve the same morphology for the compliance tensor, whose components depend upon the two anisotropic phases.
- From a mechanical point of view, R_0-orthotropic materials have a behavior somewhat different from ordinary orthotropy, e.g. the components vary like those of a second-rank tensor or are isotropic.
- Square symmetric materials share some of the remarks done for R_0-orthotropy, but geometrically speaking they are different from them and from ordinary orthotropy because they have two couples of mutually orthogonal symmetry axes tilted of $\pi/4$. This gives a periodicity of $\pi/2$ to all of the components.
- It can be seen that special orthotropies have some other interesting mechanical properties that are not possessed by ordinarily orthotropic materials.