UNIVERSITÀ DEGLI STUDI FIRENZE

UNIVERSITÀ DI PISA

Università degli Studi di Perugia

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International Doctorate in Civil and Environmental Engineering

Anisotropic Structures - Theory and Design

Strutture anisotrope: teoria e progetto

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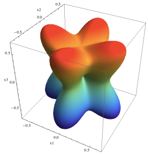


Lesson 2 - April 9, 2019 - DICEA - Universitá di Firenze

Topics of the second lesson

- Anisotropic elasticity - Part 2

Reduction of the E_{ijkl} by elastic symmetries

We consider now the effects of elastic symmetries on tensor $\mathbb{E}$; we will see that, depending upon the symmetry, some components E_{ijkl} vanish while some other can become functions of other components.

In the end, elastic symmetries reduce the number of the independent Cartesian components of $\mathbb{E}$.

It is worth to work on the C_{ij} rather than on the E_{ijkl} because simpler.

Before going on, we recall the equations that are needed in the following:

- invariance of the strain energy

$$\{\varepsilon\}^\top[C]\{\varepsilon\} = ([R]\{\varepsilon\})^\top [C][R]\{\varepsilon\} \quad \forall\{\varepsilon\} \tag{1}$$

- orthogonal tensor describing a symmetry with respect to a plane whose normal is $\mathbf{n}= (n_1, n_2, n_3)$:

$$\mathbf{U} = \mathbf{I} - 2\mathbf{n} \otimes \mathbf{n} = \begin{bmatrix} 1-2n_1^2 & -2n_1 n_2 & -2n_1 n_3 \\ & 1-2n_2^2 & -2n_2 n_3 \\ sym & & 1-2n_3^2 \end{bmatrix} \tag{2}$$

- rotation matrix corresponding, in the Kelvin's notation, to $\mathbf{U}$:

$$[R] = \begin{bmatrix} U_{11}^2 & U_{12}^2 & U_{13}^2 & \sqrt{2}U_{12}U_{13} & \sqrt{2}U_{13}U_{11} & \sqrt{2}U_{11}U_{12} \\ U_{21}^2 & U_{22}^2 & U_{23}^2 & \sqrt{2}U_{22}U_{23} & \sqrt{2}U_{23}U_{21} & \sqrt{2}U_{21}U_{22} \\ U_{31}^2 & U_{32}^2 & U_{33}^2 & \sqrt{2}U_{32}U_{33} & \sqrt{2}U_{33}U_{31} & \sqrt{2}U_{31}U_{32} \\ \sqrt{2}U_{21}U_{31} & \sqrt{2}U_{22}U_{32} & \sqrt{2}U_{23}U_{33} & U_{23}U_{32}+U_{22}U_{33} & U_{33}U_{21}+U_{31}U_{23} & U_{31}U_{22}+U_{32}U_{21} \\ \sqrt{2}U_{31}U_{11} & \sqrt{2}U_{32}U_{12} & \sqrt{2}U_{33}U_{13} & U_{32}U_{13}+U_{33}U_{12} & U_{31}U_{13}+U_{33}U_{11} & U_{31}U_{12}+U_{32}U_{11} \\ \sqrt{2}U_{11}U_{21} & \sqrt{2}U_{12}U_{22} & \sqrt{2}U_{13}U_{23} & U_{12}U_{23}+U_{13}U_{22} & U_{11}U_{23}+U_{13}U_{21} & U_{11}U_{22}+U_{12}U_{21} \end{bmatrix} \tag{3}$$

Triclinic bodies

A triclinic body has no material symmetries, so eq. (1) cannot be written $\rightarrow$ it is not possible to reduce the number of independent elastic components, that remains fixed to 21:

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & sym & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}. \tag{4}$$

Monoclinic bodies

The only symmetry of a monoclinic body is a reflection in a plane.

Without loss in generality, we can suppose to be $x_3 = 0$ the symmetry plane $\Rightarrow$ $\mathbf{n} = (0, 0, 1)$.

In such a case it is, see eqs. (2) and (3),

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow [R] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (5)$$

that applied to eq. (1) gives the condition

$$\begin{aligned}&C_{14}\varepsilon_1\varepsilon_4 + C_{24}\varepsilon_2\varepsilon_4 + C_{34}\varepsilon_3\varepsilon_4 + C_{15}\varepsilon_1\varepsilon_5 + \\ &C_{25}\varepsilon_2\varepsilon_5 + C_{35}\varepsilon_3\varepsilon_5 + C_{46}\varepsilon_4\varepsilon_6 + C_{56}\varepsilon_5\varepsilon_6 = 0,\end{aligned} \quad \forall \boldsymbol{\varepsilon} \iff \tag{6}$$

$$C_{14} = C_{24} = C_{34} = C_{15} = C_{25} = C_{35} = C_{46} = C_{56} = 0. \tag{7}$$

Hence, a monoclinic body depends upon only 13 distinct elastic moduli:

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & 0 & 0 & C_{36} \\ & & & C_{44} & C_{45} & 0 \\ & sym & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}. \tag{8}$$

Orthotropic bodies

Let us now add another plane of symmetry orthogonal to the previous one, say the plane $x_2 = 0 \;\Rightarrow\; \mathbf{n} = (0, 1, 0)$.

With the same procedure, we get successively:

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow [R] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{9}$$

$$
\begin{aligned}
&(C_{14}\varepsilon_1 + C_{24}\varepsilon_2 + C_{34}\varepsilon_3 + C_{45}\varepsilon_5)\varepsilon_4 + \\
&(C_{16}\varepsilon_1 + C_{26}\varepsilon_2 + C_{36}\varepsilon_3 + C_{56}\varepsilon_5)\varepsilon_6 = 0 \quad \forall \boldsymbol{\varepsilon} \iff \\
&C_{14} = C_{24} = C_{34} = C_{45} = C_{16} = C_{26} = C_{36} = C_{56} = 0.
\end{aligned} \tag{10}
$$

So, the existence of the second plane of symmetry has added the four supplementary conditions

$$
C_{16} = C_{26} = C_{36} = C_{45} = 0 \tag{11}
$$

to the previous eight ones, reducing hence to only 9 the number of distinct elastic moduli.

Let us now suppose the existence of a third plane of symmetry, orthogonal to the previous ones, the plane $x_1 = 0 \;\Rightarrow\; \mathbf{n} = (1, 0, 0)$.

With the same procedure, we get:

$$\mathbf{U} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow [R] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad (12)$$

$$\begin{aligned} &(C_{15}\varepsilon_1 + C_{25}\varepsilon_2 + C_{35}\varepsilon_3 + C_{45}\varepsilon_4)\varepsilon_5 + \\ &(C_{16}\varepsilon_1 + C_{26}\varepsilon_2 + C_{36}\varepsilon_3 + C_{46}\varepsilon_4)\varepsilon_6 = 0 \quad \forall \boldsymbol{\varepsilon} \iff \\ &C_{15} = C_{25} = C_{35} = C_{45} = C_{16} = C_{26} = C_{36} = C_{46} = 0. \end{aligned} \quad (13)$$

Rather surprisingly, this new symmetry condition does not give any supplementary condition to those in (7) and (11).

$\Rightarrow$ the existence of 2 orthogonal planes of elastic symmetry is physically impossible: only the presence of 1 or 3 mutually orthogonal planes of symmetry is admissible.

The class of orthotropic materials is very important, because a lot of materials or structures belong to it.

An orthotropic material depends hence upon 9 distinct elastic moduli and its matrix [C] looks like

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}. \tag{14}$$

Axially symmetric bodies

There are only 4 possible cases of axial symmetries for crystals: the 2-, 3-, 4- and 6-fold axis of symmetry (say x_3).

Let us begin with a 2-fold axis of symmetry; the covering operation corresponds hence to a rotation of π about x_3 $\rightarrow$

$$\mathbf{U}=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\Rightarrow[R]=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&1\end{bmatrix},\quad(15)$$

and we can observe that $[R]$ is the same of the monoclinic case $\rightarrow$ a 2-fold axis of symmetry coincides with a plane of symmetry.

For a 3-fold axis of symmetry, the covering operation corresponds to a rotation of $2\pi/3$ about x_3 $\rightarrow$

$$\mathbf{U} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow [R] = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & -\sqrt{\frac{3}{8}} \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & \sqrt{\frac{3}{8}} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \sqrt{\frac{3}{8}} & -\sqrt{\frac{3}{8}} & 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix} \tag{16}$$

and condition (1) gives 14 conditions on the components of $[C]$:

$$\begin{aligned} & C_{16} = C_{26} = C_{34} = C_{35} = C_{36} = C_{45} = 0, \\ & C_{22} = C_{11}, \quad C_{55} = C_{44}, \quad C_{23} = C_{13}, \quad C_{24} = -C_{14}, \\ & C_{25} = -C_{15}, \quad C_{56} = \sqrt{2}C_{14}, \quad C_{46} = \sqrt{2}C_{15}, \\ & C_{66} = C_{11} - C_{12} \end{aligned} \tag{17}$$

So, there are only 7 distinct elastic moduli:

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & 0 \\ & C_{11} & C_{13} & -C_{14} & -C_{15} & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & -\sqrt{2}C_{15} \\ & sym & & & C_{44} & \sqrt{2}C_{14} \\ & & & & & C_{11} - C_{12} \end{bmatrix}. \quad (18)$$

For a 4-fold axis of symmetry, the covering operation corresponds to a rotation of $\pi/2$ about x_3 $\rightarrow$

$$\mathbf{U} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow [R] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{19}$$

The result are 14 conditions different from the (17):

$$\begin{aligned} & C_{14} = C_{24} = C_{34} = C_{15} = C_{25} = C_{35} = \\ & C_{45} = C_{36} = C_{46} = C_{56} = 0, \\ & C_{22} = C_{11}, \quad C_{55} = C_{44}, \quad C_{23} = C_{13}, \quad C_{26} = -C_{16} \end{aligned} \tag{20}$$

This gives an elastic matrix $[C]$ still depending upon only 7 distinct moduli, but different from the previous case:

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{11} & C_{13} & 0 & 0 & -C_{16} \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym & & & C_{44} & 0 \\ & & & & & C_{66} \end{bmatrix}. \tag{21}$$

The last case of 6-fold axis of symmetry has as covering operation a rotation of $\pi/3$ about x_3 $\rightarrow$

$$\mathbf{U} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow [R] = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & \sqrt{\frac{3}{8}} \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & -\sqrt{\frac{3}{8}} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\sqrt{\frac{3}{8}} & \sqrt{\frac{3}{8}} & 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix} \tag{22}$$

The result are 16 conditions:

$$\begin{aligned} &C_{14} = C_{24} = C_{34} = C_{15} = C_{25} = C_{35} = \\ &C_{45} = C_{16} = C_{26} = C_{36} = C_{46} = C_{56} = 0, \\ &C_{22} = C_{11}, \quad C_{55} = C_{44}, \quad C_{23} = C_{13}, \quad C_{66} = C_{11} - C_{12} \end{aligned} \tag{23}$$

Finally, the elastic matrix $[C]$ depends upon only 5 moduli:

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym & & & C_{44} & 0 \\ & & & & & C_{11} - C_{12} \end{bmatrix}. \tag{24}$$

Transversely isotropic bodies

A transversely isotropic body has an axis of cylindrical symmetry, i.e. the covering operation is a rotation by any angle θ.

Many materials belong to this class: timber, fiber reinforced composites, laminated steel, pack ice etc. but not crystals.

Proceeding in the usual way we get

$$\mathbf{U} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow [R] = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & \sqrt{2}cs \\ s^2 & c^2 & 0 & 0 & 0 & -\sqrt{2}cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -\sqrt{2}cs & \sqrt{2}cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix} \tag{25}$$

$c = \cos\theta,\ s = \sin\theta$

In this case we obtain exactly the same 16 conditions (23) $\Rightarrow$ elastically, the 6-fold axis of symmetry is strictly identical to an axis of cylindrical symmetry.

Hence, two such materials cannot be distinguished using only the results of tests on stress or strain energy.

This should not be surprising, because this fact is in perfect accordance with the Neumann's principle, as the 6-fold axis of symmetry is contained in the more general case of cylindrical symmetry.

Finally, eq. (24) represents also the elastic matrix of a transversely isotropic material, who has 5 distinct elastic moduli.

Isotropy

Isotropy is the complete symmetry: all the directions are equivalent.

The conditions of isotropy could be found following the usual procedure, imposing that eq. (1) is valid for any orthogonal transformation $[R]$.

However, this general approach, that can be followed using for instance the Euler angles for expressing a generic $[R]$, results to be very cumbersome and computationally heavy.

A more direct approach is the following one: for a transversely isotropic body, all the directions orthogonal to the axis of symmetry, say x_3, are equivalent.

In other words, fixing the axes of x_1 and x_2 is completely arbitrary.

We then suppose that, besides the equivalence of all the directions in the plane perpendicular to x_3, also x_1 and x_3 are equivalent $\rightarrow$

We then impose to a material described by a transversely isotropic elastic matrix, eq. (24), this further equivalence, which is described by

$$\mathbf{U} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad [R] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{26}$$

This gives 3 new conditions:

$$C_{13} = C_{12}, \quad C_{33} = C_{11}, \quad C_{44} = C_{66} \tag{27}$$

This reduces the number of distinct elastic constants from 5 to only 2:

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{11}-C_{12} & 0 & 0 \\ & sym & & & C_{11}-C_{12} & 0 \\ & & & & & C_{11}-C_{12} \end{bmatrix} \tag{28}$$

Because x_1 is any direction, all the directions of the space are equivalent.

This can be proved showing that the elastic matrix (28) is insensitive to any change of basis leaving x_2 unchanged, i.e.

$$\mathbf{U} = \begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix} \Rightarrow [R] = \begin{bmatrix} c^2 & 0 & s^2 & 0 & \sqrt{2}cs & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ s^2 & 0 & c^2 & 0 & -\sqrt{2}cs & 0 \\ 0 & 0 & 0 & c & 0 & -s \\ -\sqrt{2}cs & 0 & \sqrt{2}cs & 0 & c^2-s^2 & 0 \\ 0 & 0 & 0 & s & 0 & c \end{bmatrix} \tag{29}$$

which gives as only condition $C_{44} = C_{11} - C_{12}$, already contained in eqs. (23) and (27): nothing is added to the previous conditions $\Rightarrow$ all the directions in any meridian plane are equivalent, i.e. the body is isotropic.

There is another, more elegant and direct way to prove that an isotropic body depends upon only two distinct moduli:

- because of isotropy, the strain energy V can depend only upon the invariants of $\boldsymbol{\varepsilon}$ and not upon its Cartesian components ($[C]$ is completely invariant for an isotropic body);
- for a linearly elastic body the Green's formula $\sigma_{ij} = \dfrac{\partial V}{\partial \varepsilon_{ij}}$ and the Hooke's law $\sigma_{ij} = E_{ijkl}\varepsilon_{kl}$ impose V to be a quadratic form of $\boldsymbol{\varepsilon}$;
- then, V can only be a linear combination of the square of the first and second invariant of $\boldsymbol{\varepsilon}$:

$$V = \frac{1}{2}c_1 I_1^2 + c_2 I_2, \tag{30}$$

- with[1]

$$I_1 = \mathrm{tr}\boldsymbol{\varepsilon} = \varepsilon_{ii}, \quad I_2 = \frac{\mathrm{tr}^2\boldsymbol{\varepsilon} - \mathrm{tr}\boldsymbol{\varepsilon}^2}{2} = \frac{\varepsilon_{ii}\ \varepsilon_{jj} - \varepsilon_{ij}\ \varepsilon_{ji}}{2}. \tag{31}$$

- The third order invariant of $\boldsymbol{\varepsilon}$, i.e. $\det\boldsymbol{\varepsilon}$, cannot enter in the expression of V, because it is a cubic function of the ε_{ij}s, while V must be a quadratic function of the ε_{ij}s.

- Then,

$$V = \frac{1}{2}\left[(c_1 + c_2)\varepsilon_{ii}\ \varepsilon_{jj} - c_2\ \varepsilon_{ij}\ \varepsilon_{ji}\right] \tag{32}$$

- the two coefficients of the combination are exactly the two independent elastic moduli.

[1] $\boldsymbol{\varepsilon}^2 = \boldsymbol{\varepsilon}\boldsymbol{\varepsilon} = \varepsilon_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\ \varepsilon_{hk}\mathbf{e}_h \otimes \mathbf{e}_k = \varepsilon_{ij}\ \varepsilon_{hk}\ \mathbf{e}_j \cdot \mathbf{e}_h(\mathbf{e}_i \otimes \mathbf{e}_k) = \varepsilon_{ij}\ \varepsilon_{hk}\ \delta_{jh}(\mathbf{e}_i \otimes \mathbf{e}_k) \to \mathrm{tr}\boldsymbol{\varepsilon}^2 = \varepsilon_{ij}\ \varepsilon_{hk}\ \delta_{jh}\mathrm{tr}(\mathbf{e}_i \otimes \mathbf{e}_k) = \varepsilon_{ij}\ \varepsilon_{hk}\ \delta_{jh}\delta_{ik} = \varepsilon_{ij}\ \varepsilon_{ji}$.

Some remarks about elastic symmetries

The results for $[C]$ are completely valid also for $[S]$; this is not the case with the Voigt's notation, where for some symmetries, not all the S_{ij} have the same expression of the corresponding C_{ij}.

Typically, some coupling components disappear in a symmetry basis. The case of orthotropic bodies is emblematic: in the orthotropic frame, the skyline of $[C]$ is exactly the same of an isotropic body and the only coupling is the Poisson's effect.

Nevertheless, this is no longer true in any other basis: in a generic basis, all the anisotropic materials, regardless of their symmetries, behave like a triclinic body, i.e. they have all the coupling terms (generally speaking, their elastic matrix is complete, none of its terms vanishes).

The only exception to this fact is isotropy; in fact, for an isotropic body the matrices $[C]$ and $[S]$ are completely invariant, i.e. their only two distinct moduli are tensor invariants and the only possible coupling is the Poisson's effect.

This is the obvious consequence of the fact that all the directions of the space are equivalent.

Physically, the fact that the least number of independent elastic constants is two means that in a stressed elastic body there are, in general, at least two distinct and independent deformation effects: the direct one and the Poisson's effect.

Elasticity of crystals and elastic syngonies

Crystals have an elastic behavior that belongs to one of the cases above or is a combination of these cases.

Examining their cases, allows us for entirely defining the 10 elastic syngonies introduced above.

In particular, referring to the Voigt's classification

1. classes 1 and 2 belong to the triclinic case, with 21 constants; their matrix $[C]$ is like in eq. (4) and this crystal syngony corresponds to the triclinic elastic syngony;
2. classes 3, 4 and 5 belong to the monoclinic case, with 13 constants; their matrix $[C]$ is like in eq. (8) and this crystal syngony corresponds to the monoclinic elastic syngony;

3. classes 6, 7 and 8 of the orthorhombic syngony belong to the orthotropic case, with 9 constants; their matrix $[C]$ is like in eq. (14) and the orthorhombic syngony corresponds hence entirely to the orthotropic elastic syngony;
4. classes 12 and 13 of the trigonal syngony belong to the 3-fold rotational symmetry case, with 7 constants; they have a matrix $[C]$ as in eq. (18) and they constitute the trigonal elastic syngony with 7 constants;
5. classes 17, 18 and 20 of the tetragonal syngony belong to the 4-fold rotational symmetry case, with 7 constants; their matrix is like in eq. (21) and they constitute the tetragonal elastic syngony with 7 constants;

6. classes 9, 10 and 11 of the trigonal syngony are a combination of the 3-fold rotational symmetry and the monoclinic symmetry cases: if the plane of symmetry is the plane $x_1 = 0$, then the usual procedure applied to the matrix (18) gives $C_{15} = 0$, and matrix (18) becomes

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ & C_{11} & C_{13} & -C_{14} & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym & & & C_{44} & \sqrt{2}C_{14} \\ & & & & & C_{11} - C_{12} \end{bmatrix}; \qquad (33)$$

If it is $x_2 = 0$ the plane of symmetry, then it is $C_{14} = 0$ and matrix (18) becomes

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\ & C_{11} & C_{13} & 0 & -C_{15} & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & -\sqrt{2}C_{15} \\ & sym & & & C_{44} & 0 \\ & & & & & C_{11} - C_{12} \end{bmatrix}; \qquad (34)$$

these cases constitute the trigonal elastic syngony with 6 constants;

7. classes 14, 15, 16 and 19 of the tetragonal syngony are a particular case of the orthotropic symmetry: they have identical elastic properties along the axis x_1 and x_2, which gives the three supplementary conditions $C_{22} = C_{11},\ C_{23} = C_{13},\ C_{55} = C_{44}$, so reducing matrix (14) to

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym & & & C_{44} & 0 \\ & & & & & C_{66} \end{bmatrix}; \qquad (35)$$

these cases constitute the tetragonal elastic syngony with 6 constants;

8. classes of the hexagonal syngony, from the 21 to the 27, belong to the 6-fold rotational symmetry, with 5 constants; together with transversely isotropic materials, that do not exist as crystals, they form the axially-symmetric elastic syngony, with $[C]$ as in eq. (24);
9. classes of the cubic syngony, from the 28 to the 32, are a particular case of the orthotropic symmetry: they have identical properties along the three axes, which gives the six supplementary conditions $C_{33} = C_{22} = C_{11},\ C_{23} = C_{13} = = C_{12},\ C_{66} = C_{55} = C_{44}$, so reducing matrix (14) to

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym & & & C_{44} & 0 \\ & & & & & C_{44} \end{bmatrix}; \qquad (36)$$

the cubic crystal syngony corresponds entirely with the cubic elastic syngony;

10. the last elastic syngony is the isotropic elastic syngony; of course, no crystal syngonies belong to this case; nevertheless, a huge number of materials have an isotropic behavior.

    Though the texts on crystals and anisotropy usually forget to consider the isotropic case, this one actually exists and for the sake of completeness we prefer here to consider it as an elastic syngony; the isotropic matrix (28) can be obtained as a particular case of the cubic one, (36), when $C_{44} = C_{11} - C_{12}$.

The technical constants of elasticity

In practical applications, engineers usually prefer to replace the use of the elastic stiffness matrix components by the so-called technical elasticity constants or engineer moduli.

Technical constants quantify an effect, a direct or a coupling one, whose mechanical meaning is immediate and that can be easily identified and measured in simple laboratory tests, like for instance unidirectional tension tests.

Of course, the set of technical constants must be equivalent to the set of independent elastic moduli:

- the number of technical constants and distinct elastic moduli must be the same, i.e. 21
- the technical constants must represent all the mechanical effects in a stressed body

The Young's moduli

The three Young's moduli generalize to anisotropy the analogous isotropic modulus and are defined in a similar way:

$$E_i := \frac{\sigma_i}{\varepsilon_i}, \quad i = 1, 2, 3, \quad \sigma_i \neq 0, \ \sigma_j = 0 \text{ for } j \neq i, \ j = 1, ..., 6. \quad (37)$$

As a consequence, from the Hooke's inverse law we get the relations (no summation over dummy indexes)

$$S_{ii} = Z_{iiii} = \frac{1}{E_i}, \quad i = 1, 2, 3. \quad (38)$$

The Young's moduli measure the extension stiffness along the direction of one of the frame axes.

Generally speaking, the three Young's moduli are different, i.e. in anisotropy the directions of the space have different stiffnesses.

The shear moduli

Also in this case, the three shear moduli generalize to anisotropy the isotropic concept of shear modulus:

$$
\begin{aligned}
&G_{ij} := \frac{\sigma_k}{2\varepsilon_k}, \quad i,j = 1,2,3, \;\; k = 4,5,6, \\
&\sigma_k \neq 0, \;\; \sigma_h = 0 \text{ for } h \neq k, \;\; h = 1,...,6.
\end{aligned}
\tag{39}
$$

To remark the discrepancy in the nomenclature of the $G_{ij}s$: the Kelvin notation is used for σ_k and ε_k but in G_{ij} the suffixes are those indicating the directions.

As a consequence, from the Hooke's inverse law we get the relations (no summation over dummy indexes)

$$
2S_{kk} = 4Z_{ijij} = \frac{1}{G_{ij}}, \quad i = 1,2,3, \;\; k = 4,5,6. \tag{40}
$$

The mechanical meaning of the G_{ij} is completely analogous to that of the Young's moduli, but it concerns shear.

Poisson's coefficients

The definition of the Poisson's coefficients or ratios in anisotropy is quite similar to isotropy:

$$\nu_{ij} := -\frac{\varepsilon_j}{\varepsilon_i}, \quad i,j = 1,2,3, \ \sigma_i \neq 0, \ \sigma_h = 0 \text{ for } h \neq i, \ h = 1,...,6. \tag{41}$$

Like for shear moduli, the nomenclature makes use of the Kelvin's notation along with the tensorial one.
From the Young's moduli definition, eq. (37), we get

$$\varepsilon_j = -\nu_{ij}\varepsilon_i = -\nu_{ij}\frac{\sigma_i}{E_i}, \quad i,j = 1,2,3. \tag{42}$$

Through the Hooke's inverse law this gives (no summation over dummy indexes)

$$S_{ji} = Z_{jjii} = -\frac{\nu_{ij}}{E_i} \quad \Rightarrow \quad \nu_{ij} = -\frac{S_{ji}}{S_{ii}}, \quad i,j = 1,2,3. \tag{43}$$

Finally, the symmetry of matrix $[S]$, consequence of the major symmetries of $\mathbb{Z}$, gives the reciprocity relations

$$\frac{\nu_{ij}}{E_i}=\frac{\nu_{ji}}{E_j},\quad i,j=1,2,3, \tag{44}$$

which reduce the number of distinct Poisson's coefficients from 6 to only 3.

Some remarks about the Poisson's coefficients:

- they measure the Poisson's effect, i.e. the deformation in a direction transversal to that of the normal stress
- generally speaking, $\nu_{12}\neq\nu_{12}\neq\nu_{23}\Rightarrow$ the Poisson's effect is different in the different directions
- because the $\nu_{ij}s$ depend upon the direction, it is possible that in some directions $\nu_{ij}\leq 0$
- some authors exchange the place of suffixes i and j in the definition of ν_{ij}

Chentsov's coefficients

The Chentsov's coefficients $\mu_{ij,kl}$ play for shear the same role of the Poisson's coefficients:

$$
\begin{aligned}
&\mu_{ij,kl} := \frac{\varepsilon_{ij}}{\varepsilon_{kl}}, \quad i,j,k,l = 1,2,3, \ i \neq j, \ k \neq l, \\
&\sigma_{kl} \neq 0, \ \sigma_{pq} = 0 \text{ for } pq \neq kl, \ p,q = 1,2,3.
\end{aligned} \tag{45}
$$

$\mu_{ij,kl}$ measures the Chentsov's effect in the plane ij due to the shear stress σ_{kl}, i.e. the ratio between the shear strain components ε_{ij} and ε_{kl}.

By the definition of the G_{ij}s, eq. (39), it follows that (no summation over dummy indexes)

$$
\varepsilon_{ij} = \mu_{ij,kl}\varepsilon_{kl} = \mu_{ij,kl}\frac{\sigma_{kl}}{2G_{kl}} \quad i,j,k,l = 1,2,3, \tag{46}
$$

and through the Hooke's inverse law we get

$$2S_{pq} = 4Z_{ijkl} = \frac{\mu_{ij,kl}}{G_{kl}} \;\Rightarrow\; \mu_{ij,kl} = \frac{S_{pq}}{S_{qq}},\ i,j,k,l = 1,2,3,\ p,q = 4,5,6, \tag{47}$$

with p that corresponds to the couple ij and q to kl according to the scheme $ii \to i\ \forall i = 1,2,3; 12 \to 6, 13 \to 5, 23 \to 4$.

The symmetry of $[S]$ gives the reciprocity relations

$$\frac{\mu_{ij,kl}}{G_{kl}} = \frac{\mu_{kl,ij}}{G_{ij}}, \tag{48}$$

that, along with the minor symmetries of $\boldsymbol{\sigma}$ and ε reduce to only 3 the number of distinct Chentsov's coefficients.

Finally, the remarks done for the ν_{ij}s can be rephrased *verbatim* for the $\mu_{ij,kl}$s.

Coefficients of mutual influence of the first type

They characterize the normal strain ε_{ii} due to the shear σ_{jk} (no summation over dummy indexes):

$$\begin{aligned}&\eta_{i,jk} := \frac{\varepsilon_{ii}}{2\varepsilon_{jk}} \quad i,j,k=1,2,3,\ j \neq k,\ \sigma_{jk} \neq 0,\\ &\sigma_{pq} = 0 \text{ for } pq \neq jk,\ p,q=1,2,3.\end{aligned} \tag{49}$$

By the definition of the G_{ij}s, eq. (39), it follows that

$$\varepsilon_{ii} = 2\eta_{i,jk}\varepsilon_{jk} = \eta_{i,jk}\frac{\sigma_{jk}}{G_{jk}}, \tag{50}$$

and through the Hooke's inverse law we get

$$\sqrt{2}S_{ip} = 2Z_{iijk} = \frac{\eta_{i,jk}}{G_{jk}} \ \Rightarrow\ \eta_{i,jk} = \frac{S_{ip}}{\sqrt{2}S_{pp}},\ i,j,k=1,2,3,\ p=4,5,6, \tag{51}$$

p corresponds to the couple jk according to the usual rule.

For the symmetry of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, the exchange of suffixes j and k has no effects, so the number of distinct coefficients is only 9.

Coefficients of mutual influence of the second type

They characterize the shear strain ε_{ij} due to the normal stress σ_{kk} (no summation over dummy indexes):

$$\eta_{ij,k} := \frac{2\varepsilon_{ij}}{\varepsilon_{kk}} \quad i,j,k = 1,2,3,\ i \neq j,\ \sigma_{kk} \neq 0,$$
$$\sigma_{pq} = 0 \text{ for } pq \neq kk,\ p,q = 1,2,3. \tag{52}$$

By the definition of the E_is, eq. (37), it follows that

$$2\varepsilon_{ij} = \eta_{ij,k}\varepsilon_{kk} = \eta_{ij,k}\frac{\sigma_{kk}}{E_k}, \tag{53}$$

and through the Hooke's inverse law we get

$$\sqrt{2}S_{pk} = 2Z_{ijkk} = \frac{\eta_{ij,k}}{E_k} \ \Rightarrow\ \eta_{ij,k} = \sqrt{2}\frac{S_{pk}}{S_{kk}},\ i,j,k = 1,2,3,\ p = 4,5,6, \tag{54}$$

p corresponds to the couple ij according to the known rule.

Like for the coefficients of the first type, the symmetries of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ reduce the number of distinct coefficients of the second type to only 9.

The coefficients of the second type are not independent from those of the first type.

In fact, the symmetry of $[S]$ gives immediately the reciprocity relations

$$\frac{\eta_{ij,k}}{E_k} = \frac{\eta_{k,ij}}{G_{ij}}, \quad i, j, k = 1, 2, 3. \tag{55}$$

So the use of the coefficients of the first or of the second type is arbitrary and equally valid.

Also for the coefficients of the first and second type can be repeated almost *verbatim* the remarks done about the other coefficients.

Some remarks about the technical constants

The relations between a technical constant and the corresponding component of $\mathbb{Z}$, given in the previous Sections, are valid regardless of the notation used, i.e. they are the same also with the Voigt's notation.

On the contrary, the relations with the components S_{ij} depends upon the notation, and those found above are not completely identical with the Voigt's notation.

It is possible, of course, to express also the components of $[C]$ as functions of the technical constants; this necessitates the inversion of $[S]$ and in the most general case it gives so complicate and long expressions that it is impossible to write them.

Nevertheless, in the important case of orthotropic materials the transformation is rather simple.

In fact, in the orthotropic frame, the inverse of matrix $[S]$, which is perfectly analogous to matrix (14), is given by

$$
\begin{aligned}
&C_{ii} = \frac{S_{jj}S_{kk} - S_{jk}^2}{S} = \frac{1 - \nu_{jk}\nu_{kj}}{\varDelta}E_i, \quad i,j,k = 1,2,3, \ i \neq j \neq k,\\
&C_{ij} = \frac{S_{ik}S_{kj} - S_{ij}S_{kk}}{S} = \frac{\nu_{ij} + \nu_{ik}\nu_{kj}}{\varDelta}E_j, \quad i,j,k = 1,2,3, \ i \neq j \neq k,\\
&C_{44} = 2G_{23}, \quad C_{55} = 2G_{31}, \quad C_{66} = 2G_{12}
\end{aligned} \tag{56}
$$

with

$$
\begin{aligned}
&S = S_{11}S_{22}S_{33} - S_{11}S_{23}^2 - S_{22}S_{13}^2 - S_{33}S_{12}^2 + 2S_{12}S_{23}S_{13},\\
&\varDelta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{32}\nu_{21}\nu_{13}.
\end{aligned} \tag{57}
$$

It is also worth to specify these results for the isotropic case

$$
[C] = \begin{bmatrix}
\frac{(1-\nu)E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\
 & \frac{(1-\nu)E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\
 & & \frac{(1-\nu)E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\
 & & & \frac{E}{1+\nu} & 0 & 0 \\
 & sym & & & \frac{E}{1+\nu} & 0 \\
 & & & & & \frac{E}{1+\nu}
\end{bmatrix}, \quad (58)
$$

$$
[S] = \begin{bmatrix}
\frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
 & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
 & & \frac{1}{E} & 0 & 0 & 0 \\
 & & & \frac{1+\nu}{E} & 0 & 0 \\
 & sym & & & \frac{1+\nu}{E} & 0 \\
 & & & & & \frac{1+\nu}{E}
\end{bmatrix}. \quad (59)
$$

To remark that with the Voigt's notation one should have $E/2(1+\nu)$ in place of $E/(1+\nu)$ for C_{44}, C_{55} and C_{66}, as well as $2(1+\nu)/E$ for S_{44}, S_{55} and S_{66}.

Bounds on the elastic constants

Elastic constants are bounded because of the physical fact that the deformation of an elastic body Ω cannot produce energy: the overall work $\mathcal{L}_{ext}$ done by the applied forces must be positive.

From the Clapeyron's Theorem

$$\mathcal{L}_{ext} = 2V = 2\left(\frac{1}{2}\int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} \; d\Omega\right), \tag{60}$$

we get the condition that the strain energy V must be positive.

Assuming the strain as independent field over Ω, then the overall condition is

$$V = \frac{1}{2}\int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} \; d\Omega > 0 \quad \forall \boldsymbol{\varepsilon} \neq \mathbf{O}. \tag{61}$$

This constraint on the deformation of an elastic body is a strong condition.

By a procedure of limit towards small volumes, it is easy to see that it must be true also locally, i.e. $\forall p \in \Omega$.

The local form of (61) injected into the Hooke's law gives the condition

$$dV = \frac{1}{2}\boldsymbol{\sigma} \cdot \varepsilon = \frac{1}{2}\varepsilon \cdot \mathbb{E}\varepsilon > 0 \quad \forall \varepsilon \neq \mathbf{O}, \tag{62}$$

from which the bounds on the elastic constants can be obtained.

Eq. (62) is the mathematical condition corresponding to the thermodynamical fact that no energy can be produced deforming an elastic body: the elasticity stiffness tensor $\mathbb{E}$ must be positive definite.

If the $\boldsymbol{\sigma}$ is taken as independent field over Ω in place of ε, we get a similar restriction on the stress energy and finally the condition that the elasticity compliance tensor $\mathbb{Z}$ must be positive definite.

Of course, the two approaches give in the end the same results for the elastic constants.

Working with the C_{ij} (the final results are easily transferred to the E_{ijkl}), we are concerned with a fundamental question: when a matrix is positive definite?

Positive definiteness of the elastic matrices

Using $[C]$, condition (62) becomes

$$\frac{1}{2}\{\varepsilon\}^\top[C]\{\varepsilon\} > 0 \;\; \forall\{\varepsilon\} \neq \{0\}, \tag{63}$$

stating the positive definiteness of matrix $[C]$.

Mathematically, the problem is clear:
$[C] = [C]^\top \Rightarrow \lambda_i \in \mathbb{R}, i = 1, ..., 6$ (Spectral Theorem) $\rightarrow$ and

$$\frac{1}{2}\{\varepsilon\}^\top[C]\{\varepsilon\} > 0 \; \forall\{\varepsilon\} \neq \{0\} \iff \lambda_i > 0 \; \forall i = 1, ..., 6. \tag{64}$$

The above result is almost useless (the Laplace's equation is of degree 6!): no analytic expression of the λ_i can be get $\rightarrow$ no bounds on the C_{ij}!.

Nevertheless, a first qualitative result is that the number of conditions to be put on the C_{ij}s is 6.

As the distinct components are, in the most general case, 21, the conditions on the C_{ij}s are not necessarily simple bounds but at least some of them are necessarily relations among some of the components.

Also, for the hexagonal, cubic and isotropic syngonies the number of conditions is redundant with respect to the distinct elastic constants $\Rightarrow$ some of them have lower and upper bounds and/or some of the bounds are redundant (this, anyway, can be true also for other syngonies).

An approach different from that using the eigenvalues must be followed; there are two possibilities: a mathematical, using an almost unknown theorem, and a mechanical one.

The mathematical one first!

A (rather unknown) mathematical approach

This approach is completely general and feasible. We need to introduce the following definitions and theorems of matrix algebra.

A principal minor of a matrix $[A]$ is the determinant of the sub-matrix extracted from $[A]$ removing an equal number of rows and columns having the same indices, i.e. preserving the leading diagonal.

A leading principal minor of order r is the determinant of a principal $r \times r$ sub-matrix whose rows and columns are the first r rows and columns of $[A]$.

Hence, a $n \times n$ matrix has n leading principal minors.

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{65}$$

We need the following two theorems:

Theorem (necessary condition for a symmetric matrix to be positive definite)

All the principal minors of a positive definite $n \times n$ symmetric matrix $[A]$ are positive.

Theorem (necessary and sufficient condition for a symmetric matrix to be positive definite)

For a $n \times n$ symmetric matrix $[A]$ to be positive definite it is necessary and sufficient that its n leading principal minors are all positive.

The six principal minors of $[C]$ are

$$M_1 = C_{11}, \quad M_2 = \begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{vmatrix}, \quad M_3 = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{vmatrix},$$

$$M_4 = \begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{12} & C_{22} & C_{23} & C_{24} \\ C_{13} & C_{23} & C_{33} & C_{34} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{vmatrix}, \quad M_5 = \begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} \end{vmatrix},$$

$$M_6 = \det[C]. \tag{66}$$

Contrarily to the eigenvalues, it is always possible to explicit the above expressions and hence the 6 conditions

$$M_i > 0,\ \ i = 1, ..., 6. \tag{67}$$

That is why the use of Theorem 2 is more interesting than condition (64), though to write down the 6 conditions in the most general case of a triclinic material gives very long expressions.

Simpler expressions can be obtained for different elastic syngonies (redundant bounds are omitted):

- orthotropic elastic syngony, eq. (14):

$$\begin{aligned}
&C_{ii} > 0, \quad i = 1, 4, 5, 6,\\
&C_{11}C_{22} - C_{12}^2 > 0,\\
&C_{11}C_{22}C_{33} - C_{33}C_{12}^2 - C_{11}C_{23}^2-\\
&C_{22}C_{13}^2 + 2C_{12}C_{13}C_{23} > 0;
\end{aligned} \tag{68}$$

- tetragonal elastic syngony with 6 constants, eq. (35):

$$\begin{aligned}
&C_{44} > 0,\\
&C_{66} > 0,\\
&C_{11}^2 - C_{12}^2 > 0,\\
&(C_{11} - C_{12})\left[C_{33}(C_{11} + C_{12}) - 2C_{13}^2\right] > 0;
\end{aligned} \tag{69}$$

- axially symmetric elastic syngony, eq. (24):

$$\begin{aligned}
&C_{44} > 0,\\
&C_{11}^2 - C_{12}^2 > 0,\\
&(C_{11} - C_{12})\left[C_{33}(C_{11} + C_{12}) - 2C_{13}^2\right] > 0;
\end{aligned} \tag{70}$$

- cubic elastic syngony, eq. (36)

$$
\begin{aligned}
&C_{44} > 0, \\
&C_{11} - C_{12} > 0, \\
&C_{11} + 2C_{12} > 0;
\end{aligned}
\tag{71}
$$

- isotropic elastic syngony, eq. (28):

$$
\begin{aligned}
&C_{11} - C_{12} > 0, \\
&C_{11} + 2C_{12} > 0.
\end{aligned}
\tag{72}
$$

A (better known) mechanical approach

This method is based upon the fact that V must be positive for each possible choice of the strain field ε.

This allows for choosing particularly simple strain fields, giving some direct, simple results. Let us see how (no summation over dummy indexes).

Chose a field $\{\varepsilon\}$ with only one component $\varepsilon_i \neq 0$. Then,

$$dV > 0 \iff C_{ii} > 0, \ i = 1, ..., 6; \tag{73}$$

we get hence 6 necessary conditions, so they do not constitute a set of necessary and sufficient conditions for the positiveness of V.

Nevertheless, they give us a precious information: all the moduli responsible for the direct effects are strictly positive.

Using the stress energy instead of the strain energy, it is immediately recognized that it is also:

$$S_{ii} > 0 \ \forall i = 1, ..., 6. \tag{74}$$

Bounds on the technical constants

The results of eqs. (38), (40) and (74) give immediately

$$E_i > 0, \quad G_{ij} > 0 \ \ \forall i, j = 1, 2, 3 : \tag{75}$$

all the Young's and shear moduli are strictly positive quantities, result that is valid for any kind of elastic syngony.

To these necessary conditions some other relations for the technical constants can be added.

First of all, let us consider a spherical state of stress; it is then easy to see that

$$\begin{gathered}\{\sigma\} = \sigma\{I\} \quad \Rightarrow \quad \{\sigma\}^\top[S]\{\sigma\} > 0 \quad \iff \\ S_{11} + S_{22} + S_{33} + 2(S_{13} + S_{32} + S_{21}) > 0.\end{gathered} \tag{76}$$

Replacing in the above result the expressions of the S_{ij}s from eqs. (38) and (43) gives the condition

$$\frac{1-2\nu_{12}}{E_1}+\frac{1-2\nu_{23}}{E_2}+\frac{1-2\nu_{31}}{E_3}>0. \tag{77}$$

This result is valid regardless of the elastic syngony; for the cubic and isotropic syngonies it becomes the well known bound $\nu<1/2$ on the Poisson's coefficient.

A simpler but rougher estimation can be obtained from bound (77) (see Lekhnitskii):

$$\frac{3-2(\nu_{12}+\nu_{23}+\nu_{31})}{\min\{E_1,E_2,E_3\}}>\frac{1-2\nu_{12}}{E_1}+\frac{1-2\nu_{23}}{E_2}+\frac{1-2\nu_{31}}{E_3}>0 \Rightarrow$$
$$\nu_{12}+\nu_{23}+\nu_{31}<\frac{3}{2}. \tag{78}$$

Some other necessary conditions can be given expressing the C_{ii} in terms of the technical parameters.

For the triclinic syngony the calculations are too complicate, while for the orthotropic syngony this is possible.

The supplementary bounds can be found expressing the (73) as functions of the technical constants through eq. (56) and taking into account the positivity of the Young's moduli, eq. (75):

$$\begin{aligned}
&1-\nu_{ij}\nu_{ji}>0 \quad \forall i,j=1,2,3;\\
&\varDelta=1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-2\nu_{32}\nu_{21}\nu_{13}>0.
\end{aligned} \tag{79}$$

Condition $(79)_2$ can be transformed to

$$\nu_{32}\nu_{21}\nu_{13}<\frac{1}{2}\left(1-\nu_{32}^2\frac{E_2}{E_3}-\nu_{21}^2\frac{E_1}{E_2}-\nu_{13}^2\frac{E_3}{E_1}\right)<\frac{1}{2}. \tag{80}$$

Through the reciprocity conditions on the Poisson's coefficients, eq. (44), conditions $(79)_1$ can be written also as

$$|\nu_{ij}| < \sqrt{\frac{E_i}{E_j}} \quad \forall i, j = 1, 2, 3, \tag{81}$$

or equivalently

$$|S_{ij}| < \sqrt{S_{ii} S_{jj}} \quad \forall i, j = 1, 2, 3. \tag{82}$$

Some remarks:

- the bounds concern frame dependent quantities, and of course they are more easily written in a frame composed by symmetry directions. Then, the only, general, necessary and sufficient conditions are the (67), that can always be written and used in numerical applications, e.g. for checking the validity of the results of experimental tests;

- in the case of orthotropic materials, a set of conditions on the technical constants can be easily written, but it is still questionable whether or not it constitutes a set of necessary and sufficient conditions for the positivity of the strain energy, a point never treated in the literature;
- bounds on the Chentsov's and mutual influence coefficients are completely unknown in the literature;
- In the case of isotropic materials, the conditions of positivity of the strain energy reduce to the well known 3 bounds on E and ν

$$E > 0, \quad -1 < \nu < \frac{1}{2} \tag{83}$$

- rather surprisingly, if the bounds are written for the two distinct components of $[C]$, C_{11} and C_{12}, the bounds are only 2, see eq. (72):

$$C_{11} - C_{12} > 0, \quad C_{11} + 2C_{12} > 0 \tag{84}$$

- the same happens when the isotropic law is written under the form of the Lamé's equations

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda \operatorname{tr}\boldsymbol{\varepsilon}\mathbf{I} : \tag{85}$$

  the only two bounds on the Lamé's constants λ and μ are

$$\mu > 0, \quad 2\mu + 3\lambda > 0, \tag{86}$$

  that corresponds exactly to bounds (84);

- this shows that the number of necessary and sufficient conditions for the strain energy to be positive depends upon the choice of the elastic constants and that, anyway, it is quite hard to establish *a priori* its value, whose maximum remains anyway 6;
- all the bounds above are written with frame dependent quantities (exception made for isotropy). In particular, conditions (68) to (71) are valid exclusively in the symmetry frame where the respective matrices $[C]$ have been written;
- we will see that for the plane case it is possible, with the polar formalism, to give completely invariant necessary and sufficient bounds, i.e. bounds established on tensor invariants, which are not yet known for the general 3D case.

An observation about the decomposition of V

Let us consider a point which is true at least for isotropic materials but often thought as generally true also for other elastic syngonies: is it possible to decompose the strain energy into *spherical* and *deviatoric* parts?

In other words, we ponder whether or not it is always possible to write

$$V = V_{sph} + V_{dev}, \tag{87}$$

where V_{sph}, the spherical part of V is produced exclusively by the spherical part of ε and by its corresponding part of $\boldsymbol{\sigma}$, i.e.

$$V_{sph} = \frac{1}{2}\boldsymbol{\varepsilon}_{sph} \cdot \mathbb{E}\boldsymbol{\varepsilon}_{sph}, \tag{88}$$

and V_{dev}, the deviatoric part of V is produced exclusively by the deviatoric part of ε and by its corresponding part of $\boldsymbol{\sigma}$, i.e.

$$V_{dev} = \frac{1}{2}\boldsymbol{\varepsilon}_{dev} \cdot \mathbb{E}\boldsymbol{\varepsilon}_{dev}. \tag{89}$$

Mechanically, such a decomposition means that V can be considered as the sum of two parts:

- V_{sph}, due to volume changes not accompanied by shape changes
- V_{dev}, produced by isochoric shape changes

This decomposition is, for instance, at the basis of the Hüber-Hencky-von Mises criterion, where the only V_{dev} is considered to be responsible of yielding.

It is always possible to decompose $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ into a spherical and a deviatoric part

$$\begin{aligned} &\boldsymbol{\sigma} = \boldsymbol{\sigma}_{sph} + \boldsymbol{\sigma}_{dev}, \quad \boldsymbol{\sigma}_{sph} = \frac{1}{3}\mathrm{tr}\boldsymbol{\sigma}\ \mathbf{I}, \quad \boldsymbol{\sigma}_{dev} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_{sph}, \\ &\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{sph} + \boldsymbol{\varepsilon}_{dev}, \quad \boldsymbol{\varepsilon}_{sph} = \frac{1}{3}\mathrm{tr}\boldsymbol{\varepsilon}\ \mathbf{I}, \quad \boldsymbol{\varepsilon}_{dev} = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_{sph}, \end{aligned} \tag{90}$$

Any spherical part is orthogonal to any deviatoric part:

$$\boldsymbol{\sigma}_{sph} \cdot \boldsymbol{\varepsilon}_{dev} = \frac{1}{3}\mathrm{tr}\boldsymbol{\sigma}\ \mathbf{I} \cdot (\boldsymbol{\varepsilon} - \frac{1}{3}\mathrm{tr}\boldsymbol{\varepsilon}\ \mathbf{I}) = \frac{1}{3}\mathrm{tr}\boldsymbol{\varepsilon}\ \mathrm{tr}\boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}\boldsymbol{\varepsilon}\ \mathrm{tr}\boldsymbol{\sigma} = 0,$$
$$\boldsymbol{\sigma}_{dev} \cdot \boldsymbol{\varepsilon}_{sph} = (\boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}\boldsymbol{\sigma}\ \mathbf{I}) \cdot \frac{1}{3}\mathrm{tr}\boldsymbol{\varepsilon}\ \mathbf{I} = \frac{1}{3}\mathrm{tr}\boldsymbol{\varepsilon}\ \mathrm{tr}\boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}\boldsymbol{\varepsilon}\ \mathrm{tr}\boldsymbol{\sigma} = 0. \tag{91}$$

Using decomposition (90) we have

$$V = \frac{1}{2}\boldsymbol{\varepsilon} \cdot \mathbb{E}\boldsymbol{\varepsilon} = \frac{1}{2}(\boldsymbol{\varepsilon}_{sph} + \boldsymbol{\varepsilon}_{dev}) \cdot \mathbb{E}(\boldsymbol{\varepsilon}_{sph} + \boldsymbol{\varepsilon}_{dev}) =$$
$$\frac{1}{2}\boldsymbol{\varepsilon}_{sph} \cdot \mathbb{E}\boldsymbol{\varepsilon}_{sph} + \frac{1}{2}\boldsymbol{\varepsilon}_{dev} \cdot \mathbb{E}\boldsymbol{\varepsilon}_{dev} + \frac{1}{2}\boldsymbol{\varepsilon}_{sph} \cdot \mathbb{E}\boldsymbol{\varepsilon}_{dev} + \frac{1}{2}\boldsymbol{\varepsilon}_{dev} \cdot \mathbb{E}\boldsymbol{\varepsilon}_{sph} \tag{92}$$

Decomposition (87) is true $\iff$

$$\boldsymbol{\varepsilon}_{sph} \cdot \mathbb{E}\boldsymbol{\varepsilon}_{dev} = 0 \quad \Rightarrow \quad \mathrm{tr}\left[\boldsymbol{\varepsilon}_{sph}^{\top}(\mathbb{E}\boldsymbol{\varepsilon}_{dev})\right] = 0 \ \ \forall \boldsymbol{\varepsilon}. \tag{93}$$

In fact, whenever eq. (93) is satisfied, for the definition of $\mathbb{E}^\top$ it is

$$\varepsilon_{dev} \cdot \mathbb{E}\varepsilon_{sph} = \mathbb{E}^\top\varepsilon_{dev} \cdot \varepsilon_{sph} = \varepsilon_{sph} \cdot \mathbb{E}\varepsilon_{dev}, \tag{94}$$

because $\mathbb{E} = \mathbb{E}^\top$. This result shows that the two mixed terms in (92) are identical.

Through (90), condition (93) can be written as

$$\mathrm{tr}\left[\frac{1}{3}\mathrm{tr}\varepsilon\ \mathbf{I}(\mathbb{E}\varepsilon_{dev})\right] = 0 \quad \forall\varepsilon \quad \iff \quad \mathrm{tr}(\mathbb{E}\varepsilon_{dev}) = 0. \tag{95}$$

The components of $\mathbb{E}$ must satisfy eq. (95) for the decomposition (87) to be possible. It can be rewritten as

$$\mathrm{tr}\left[\mathbb{E}\left(\varepsilon - \frac{1}{3}\mathrm{tr}\varepsilon\ \mathbf{I}\right)\right] = 0 \;\Rightarrow\; 3\mathrm{tr}(\mathbb{E}\varepsilon) - \mathrm{tr}\varepsilon\ \mathrm{tr}(\mathbb{E}\mathbf{I}) = 0 \;\; \forall\varepsilon. \tag{96}$$

Actually, condition (93) corresponds to impose that

$$\boldsymbol{\sigma}_{dev} = \mathbb{E}\boldsymbol{\varepsilon}_{dev}, \quad \boldsymbol{\sigma}_{sph} = \mathbb{E}\boldsymbol{\varepsilon}_{sph}, \tag{97}$$

as it can be easily recognized.

Condition (96) can be written by components:

$$E_{hhkk}\varepsilon_{ii} - 3E_{jjpq}\varepsilon_{pq} = 0 \;\; \forall\varepsilon_{mn}, \quad i, j, h, k, p, q, m, n = 1, 2, 3. \tag{98}$$

Generally speaking, this quantity does not vanish for any possible choice of $\boldsymbol{\varepsilon}$.

As a consequence, for a generic anisotropic material decomposition of the strain energy into a spherical and deviatoric part is not possible.

Nevertheless, it can be checked that for the cubic syngony eq. (98) is always satisfied.

In fact, for an orthotropic material condition (98) becomes

$$
\begin{aligned}
&\frac{1}{3}[E_{1111}\left(2\varepsilon_{11}-\varepsilon_{22}-\varepsilon_{33}\right)+E_{2222}\left(2\varepsilon_{22}-\varepsilon_{11}-\varepsilon_{33}\right)+\\
&E_{3333}\left(2\varepsilon_{33}-\varepsilon_{22}-\varepsilon_{11}\right)]+\\
&\frac{2}{3}[E_{1122}\left(\varepsilon_{11}+\varepsilon_{22}-2\varepsilon_{33}\right)+E_{1133}\left(\varepsilon_{11}+\varepsilon_{33}-2\varepsilon_{22}\right)+\\
&E_{2233}\left(\varepsilon_{22}+\varepsilon_{33}-2\varepsilon_{11}\right)]=0,
\end{aligned}
\tag{99}
$$

condition which is not yet satisfied, generally speaking, but which is always satisfied when

$$
E_{1111}=E_{2222}=E_{3333}, \quad E_{1122}=E_{2233}=E_{1133}, \tag{100}
$$

i.e. by cubic materials and *a fortiori* by isotropic materials.

Determination of symmetry planes

The classification in elastic syngonies presupposes that, for a given material, the existing equivalent directions are known, so as to write $\mathbb{E}$, or equivalently $[C]$, in a symmetry frame.

But when a material is completely unknown, the independent measures to be done in experimental tests to characterize the material are as much as 21 (impossible!), $[C]$ is a full matrix and the possible symmetry planes remains unknown.

The problem is hence: given a general matrix $[C]$, is it possible to determine if some planes of symmetry exist and which they are?

We will see that in 2D it is very simple to determine the symmetry directions using the polar formalism.

In the 3D case, the problem is much more complicate; it has been solved by Cowin and Mehrabadi in two works (1987-89), successively completed by Ting (1996).

We give here a brief account of these results.

Be $\mathbf{n}$ and $\mathbf{m} \in \mathcal{V}$, $|\mathbf{n}| = |\mathbf{m}| = 1$, $\mathbf{m} \cdot \mathbf{n} = 0$, with $\mathbf{n}$ orthogonal to a symmetry plane for a material whose elastic tensor is $\mathbb{E}$.

Consider the following second-rank symmetric tensors: $\mathbf{V} = \mathbb{E}\mathbf{I}$, $\mathbf{W}$ the acoustic[2] tensor relative to the basis direction $\mathbf{e}_p$, $\mathbf{X}$ and $\mathbf{Y}$ the acoustic tensors relative to $\mathbf{n}$ and $\mathbf{m}$, respectively.[3] Then:

Theorem

The following statements are equivalent ($\lambda_i \in \mathbb{R}$, $i = 1, ..., 6$):

1. *the material has a plane of symmetry whose normal is $\mathbf{n}$;*
2. $\mathbf{V}\mathbf{n} = \lambda_1 \mathbf{Y}\mathbf{n} = \lambda_2 \mathbf{n}$;
3. $\mathbf{W}\mathbf{n} = \lambda_3 \mathbf{Y}\mathbf{n} = \lambda_4 \mathbf{n}$;
4. $\mathbf{X}\mathbf{n} = \lambda_5 \mathbf{Y}\mathbf{n} = \lambda_6 \mathbf{n}$.

---

[2] The *acoustic* or *Green-Christoffel tensor* $\mathbf{A}_\mathbf{u}$ relative to the direction $\mathbf{u}$ is the unique tensor such that $\mathbf{A}_\mathbf{u}\mathbf{w} = \mathbb{E}(\mathbf{w} \otimes \mathbf{u})\mathbf{u}\ \forall \mathbf{w} \in \mathcal{V}$.

[3] It is simple to verify that

$$\mathbf{V} = \mathbb{E}\mathbf{I} = E_{ikqq}\mathbf{e}_i \otimes \mathbf{e}_k,\ \mathbf{W} = E_{ipkp}\mathbf{e}_i \otimes \mathbf{e}_k,\ \mathbf{X} = E_{ilkm} n_l n_m \mathbf{e}_i \otimes \mathbf{e}_k,\ \mathbf{Y} = E_{ijkh} m_j m_h \mathbf{e}_i \otimes \mathbf{e}_k.$$

Proof. Without loss of generality, let us suppose that $\mathbf{n} = \mathbf{e}_1$ and $\mathbf{m} = \cos\theta \mathbf{e}_2 + \sin\theta \mathbf{e}_3$.

When $\mathbf{n}$ is an eigenvector of $\mathbf{V}$, $\mathbf{W}$, $\mathbf{X}$ or $\mathbf{Y}$ then

$$\begin{aligned}
&\mathbf{V}\mathbf{n} = \lambda_v \mathbf{n} \;\rightarrow\; E_{i1qq}\mathbf{e}_i = \lambda_v \mathbf{e}_1, \\
&\mathbf{W}\mathbf{n} = \lambda_w \mathbf{n} \;\rightarrow\; E_{ip1p}\mathbf{e}_i = \lambda_w \mathbf{e}_1, \\
&\mathbf{X}\mathbf{n} = \lambda_x \mathbf{n} \;\rightarrow\; E_{i111}\mathbf{e}_i = \lambda_x \mathbf{e}_1, \\
&\mathbf{Y}\mathbf{n} = \lambda_y \mathbf{n} \;\rightarrow \\
&[E_{i212}\cos^2\theta + E_{i313}\sin^2\theta + \\
&\;(E_{i213} + E_{i312})\sin\theta\cos\theta]\mathbf{e}_i = \lambda_y \mathbf{e}_1 \;\; \forall\theta.
\end{aligned} \tag{101}$$

For $i = 1$, the above results give the values of the respective eigenvalues, but for $i = 2, 3$ we get, respectively,

$$
\begin{aligned}
&E_{21qq} = E_{31qq} = 0,\\
&E_{2p1p} = E_{3p1p} = 0,\\
&E_{2111} = E_{3111} = 0,\\
&E_{2212}\cos^2\theta + E_{2313}\sin^2\theta + (E_{2213} + E_{2312})\sin\theta\cos\theta =\\
&E_{3212}\cos^2\theta + E_{3313}\sin^2\theta + (E_{3213} + E_{3312})\sin\theta\cos\theta = 0\ \forall\theta.
\end{aligned}
\tag{102}
$$

Passing to the C_{ij}s (for the sake of convenience)

$$
\begin{aligned}
&C_{15} + C_{25} + C_{35} = C_{16} + C_{26} + C_{36} = 0,\\
&C_{15} + C_{35} + \frac{C_{46}}{\sqrt{2}} = C_{16} + C_{26} + \frac{C_{45}}{\sqrt{2}} = 0,\\
&C_{15} = C_{16} = 0,\\
&C_{25} = C_{26} = C_{35} = C_{36} = C_{45} = C_{46} = 0.
\end{aligned}
\tag{103}
$$

If the material has $x_1 = 0$ as unique plane of symmetry, it belongs to the monoclinic syngony and its matrix $[C]$ is

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ & C_{22} & C_{23} & C_{24} & 0 & 0 \\ & & C_{33} & C_{34} & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}, \tag{104}$$

that is:

$$C_{15} = C_{16} = C_{25} = C_{26} = C_{35} = C_{36} = C_{45} = C_{46} = 0 \tag{105}$$

It is then clear that conditions $(103)_{1,4}$, $(103)_{2,4}$ or $(103)_{3,4}$ imply (105) and vice-versa.$\square$

This theorem states that the material has a plane of symmetry whose normal is **n** if and only if **n** is the eigenvector of **Y** and of at least another tensor among **V**, **W** or **X**.

Physical interpretations

A physical interpretation of Theorem 3 is possible in the frame of the acoustics theory:

- $\mathbf{X}$ is the acoustic tensor for the elastic waves that propagate in the direction of $\mathbf{n}$
- an elastic wave is a longitudinal wave whenever $\mathbf{n}$ is an eigenvector of $\mathbf{X}$
- in such a case, $\mathbf{n}$ is called a specific direction of $\mathbf{X}$
- in an anisotropic material there exist always at least 3 different specific directions (Kolodner, 1966)
- when $\mathbf{n}$ is an eigenvector of $\mathbf{Y}$, then the wave is transversal, $\mathbf{m}$ is the direction of the wave propagation and $\mathbf{n}$ is called the specific axis
- then conditions $(103)_{3,4}$, i.e. when $\mathbf{n}$ is an eigenvector of $\mathbf{X}$ and $\mathbf{Y}$, are equivalent to say that $\mathbf{n}$ is at the same time a specific direction and a specific axis.

A statical interpretation has also been given by Hayes and Norris (1991).

It traduces the above acoustics conditions into equivalent statical conditions.

They have been resumed in the following

Theorem

A material has a plane of symmetry if and only if at least two orthogonal planes of pure shear exist, sharing a common shear direction which is the normal to the plane of symmetry.

Curvilinear anisotropy

When in a body there are directions that are not parallel but mechanically equivalent, then the body possesses a curvilinear anisotropy.

It is still possible to write the Hooke's law in a rectangular coordinate system.

However, in doing so, the components of $[C]$ or $[S]$ are no more constants, but vary with the position according to the variation of the coordinate directions with respect to the equivalent directions.

Be $\{\xi, \eta, \zeta\}$ the coordinate directions of the curvilinear coordinates that coincide with the mechanically equivalent directions. With self-evident meaning of the symbols, the Hooke's law can be written in the curvilinear coordinate system as

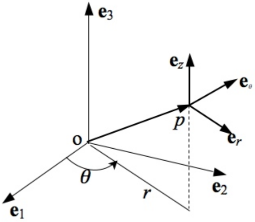
$$
\left\{\begin{array}{c}
\sigma_{\xi\xi} \\
\sigma_{\eta\eta} \\
\sigma_{\zeta\zeta} \\
\sqrt{2}\sigma_{\eta\zeta} \\
\sqrt{2}\sigma_{\zeta\xi} \\
\sqrt{2}\sigma_{\xi\eta}
\end{array}\right\} =
\left[\begin{array}{cccccc}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
 & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
 & & C_{33} & C_{34} & C_{35} & C_{36} \\
 & & & C_{44} & C_{45} & C_{46} \\
 & sym & & & C_{55} & C_{56} \\
 & & & & & C_{66}
\end{array}\right]
\left\{\begin{array}{c}
\varepsilon_{\xi\xi} \\
\varepsilon_{\eta\eta} \\
\varepsilon_{\zeta\zeta} \\
\sqrt{2}\varepsilon_{\eta\zeta} \\
\sqrt{2}\varepsilon_{\zeta\xi} \\
\sqrt{2}\varepsilon_{\xi\eta}
\end{array}\right\}, \quad (106)
$$

where the C_{ij}s are constants.

In some cases of non homogenous bodies, the C_{ij}s can depend upon the coordinates $\{\xi, \eta, \zeta\}$.

Of course, if some type of elastic symmetry is present in the body, then some of the C_{ij}s can be null, as in the ordinary cases of the elastic syngonies.

A special case of curvilinear anisotropy is that of cylindrical anisotropy: the body has an axis of symmetry, all the directions orthogonal or parallel to it are equivalent, as well as all the directions orthogonal to them.



Using a customary set of cylindrical coordinates $\{r, \theta, z\}$, with z the axis of symmetry, the Hooke's law is

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sqrt{2}\sigma_{\theta z} \\ \sqrt{2}\sigma_{zr} \\ \sqrt{2}\sigma_{r\theta} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & sym & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \sqrt{2}\varepsilon_{\theta z} \\ \sqrt{2}\varepsilon_{zr} \\ \sqrt{2}\varepsilon_{r\theta} \end{Bmatrix}. \quad (107)$$

A special case of cylindrical anisotropy is that of cylindrical orthotropy: each plane which is radial, tangential or orthogonal to the symmetry axis is a plane of symmetry.

In such a case matrix $[C]$ in eq. (107) is simplified:

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sqrt{2}\sigma_{\theta z} \\ \sqrt{2}\sigma_{zr} \\ \sqrt{2}\sigma_{r\theta} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \sqrt{2}\varepsilon_{\theta z} \\ \sqrt{2}\varepsilon_{zr} \\ \sqrt{2}\varepsilon_{r\theta} \end{Bmatrix}. \quad (108)$$

It is worth noting that cylindrical orthotropy is not equivalent to transverse isotropy (that in fact depends upon only 5 constants, not upon 9).

Actually, transverse isotropy is a special case of cylindrical orthotropy, because not only the radial and tangential directions are equivalent, but all the directions lying in a plane orthogonal to the symmetry axis are equivalent directions.

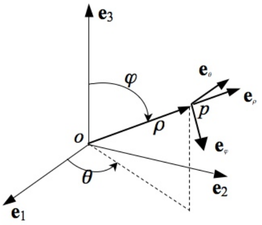
Some examples of cylindrical anisotropy are:

- some types of wood with regular yearly cylindrical layers
- metallic pipes, for their manufacturing process
- a circular reinforced concrete slab with steel bars disposed radially and circumferentially
- a bicycle wheel, when homogenized
- a circular stone arch

In spherical anisotropy there is a center of symmetry and all the rays emanating from and the tangents to parallels and meridians are equivalent directions.

Using a standard spherical coordinate systems $\{\rho, \theta, \varphi\}$, where the directions of the coordinate axes coincide with the equivalent directions, eq. (106) becomes

$$\begin{Bmatrix} \sigma_{\rho\rho} \\ \sigma_{\theta\theta} \\ \sigma_{\varphi\varphi} \\ \sqrt{2}\sigma_{\theta\varphi} \\ \sqrt{2}\sigma_{\varphi\rho} \\ \sqrt{2}\sigma_{\rho\theta} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & sym & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{\rho\rho} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{\varphi\varphi} \\ \sqrt{2}\varepsilon_{\theta\varphi} \\ \sqrt{2}\varepsilon_{\varphi\rho} \\ \sqrt{2}\varepsilon_{\rho\theta} \end{Bmatrix}. \quad (109)$$



The case of spherical orthotropy is get when each meridian and tangential plane is a plane of symmetry as well as each plane orthogonal to these two planes:

$$\left\{\begin{array}{c}\sigma_{\rho\rho}\\ \sigma_{\theta\theta}\\ \sigma_{\varphi\varphi}\\ \sqrt{2}\sigma_{\theta\varphi}\\ \sqrt{2}\sigma_{\varphi\rho}\\ \sqrt{2}\sigma_{\rho\theta}\end{array}\right\}=\left[\begin{array}{cccccc}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ & C_{22} & C_{23} & 0 & 0 & 0\\ & & C_{33} & 0 & 0 & 0\\ & & & C_{44} & 0 & 0\\ & sym & & & C_{55} & 0\\ & & & & & C_{66}\end{array}\right]\left\{\begin{array}{c}\varepsilon_{\rho\rho}\\ \varepsilon_{\theta\theta}\\ \varepsilon_{\varphi\varphi}\\ \sqrt{2}\varepsilon_{\theta\varphi}\\ \sqrt{2}\varepsilon_{\varphi\rho}\\ \sqrt{2}\varepsilon_{\rho\theta}\end{array}\right\}. \quad (110)$$

To remark the difference between isotropy and spherical orthotropy: isotropy is a special case of spherical orthotropy, because all the directions are equivalent, not only those emanating from the centre of symmetry.

This reduces the number of independent elastic constants from 9 to only 2.

Some examples of anisotropic materials

We give for some materials the matrix $[C]$ (in GPa) and the 3D-directional diagrams of some of the technical constants.

These last have been obtained as the value get by the constant on the axis of x_1' of a frame $\{x_1', x_2', x_3'\}$ rotated with respect to the frame $\{x_1, x_2, x_3\}$ where the matrix $[C]$ is known.

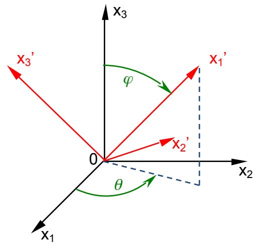


Figure: Scheme of the frame rotation for tracing the elastic constants 3D-graphics.

The rotation matrix $[R]$ is obtained through a rotation tensor $\mathbf{U}$ that is

$$\mathbf{U} = \begin{bmatrix} \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \\ -\sin\theta & \cos\theta & 0 \\ -\cos\varphi\cos\theta & -\cos\varphi\sin\theta & \sin\varphi \end{bmatrix}. \tag{111}$$

The compliance matrix $[S']$ in the rotated frame can be easily obtained:

$$\begin{gathered} \{\varepsilon\} = [S]\{\sigma\} \;\rightarrow\; [R]^\top\{\varepsilon'\} = [S][R]^\top\{\sigma'\} \;\rightarrow \\ \{\varepsilon'\} = [R][S][R]^\top\{\sigma'\} \;\Rightarrow\; [S'] = [R][S][R]^\top. \end{gathered} \tag{112}$$

Once the S'_{ij}s known, the technical constants can be easily calculated.

Through eqs. (111) and (112) it can be shown that for the materials of the hexagonal elastic syngony it is always

$$S_{14} = S_{16} = S_{24} = S_{26} = S_{34} = S_{36} = S_{45} = S_{56} = 0. \qquad (113)$$

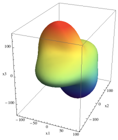
For these materials, the only Chentsov's and mutual influence coefficients that are not identically null are

$$\mu_{23,12}, \eta_{1,31}, \eta_{2,31}, \eta_{3,31}, \eta_{31,1}, \eta_{31,2}, \eta_{31,3} \qquad (114)$$

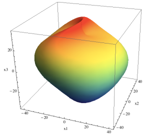
Anorthite ($CaAl_2Si_2O_8$)

Crystal syngony: Monoclinic, $N = 13$, plane of symmetry: $x_2 = 0$.

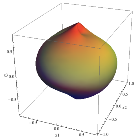
$$[C] = \begin{bmatrix} 124 & 66 & 50 & 0 & -26.9 & 0 \\ & 205 & 42 & 0 & -9.9 & 0 \\ & & 156 & 0 & -25.4 & 0 \\ & & & 48 & 0 & -2 \\ & sym & & & 80 & 0 \\ & & & & & 84 \end{bmatrix}$$



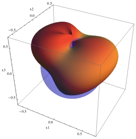
E_1



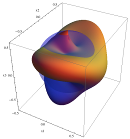
G_{12}



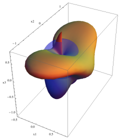
ν_{12}



$\mu_{23,12}$



$\eta_{1,31}$

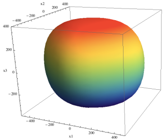


$\eta_{31,1}$

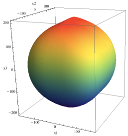
Perovskite ($CaTiO_3$)

Crystal syngony: Orthorhombic, $N = 9$.

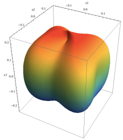
$$[C] = \begin{bmatrix} 515 & 117 & 117 & 0 & 0 & 0 \\ & 525 & 139 & 0 & 0 & 0 \\ & & 435 & 0 & 0 & 0 \\ & & & 48 & 0 & 0 \\ & sym & & & 404 & 0 \\ & & & & & 350 \end{bmatrix}$$



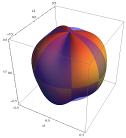
E_1



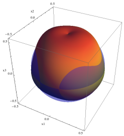
G_{12}



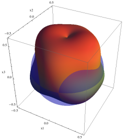
ν_{12}



$\mu_{23,12}$



$\eta_{1,31}$

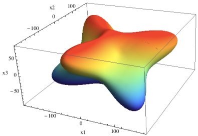


$\eta_{31,1}$

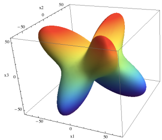
Dolomite ($CaMg(CO_3)_2$)

Crystal syngony: Trigonal, $N = 7$. (* estimated)

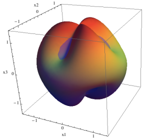
$$[C] = \left[\begin{array}{cccccc} 196.6 & 64.4 & 54.7 & 31.7 & 25.3^{*} & 0 \\ & 196.6 & 54.7 & -31.7 & -25.3^{*} & 0 \\ & & 110 & 0 & 0 & 0 \\ & & & 83.2 & 0 & -35.84 \\ & sym & & & 83.2 & 44.8 \\ & & & & & 132.2 \end{array}\right]$$



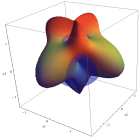
E_1



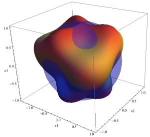
G_{12}



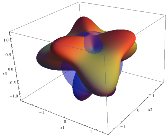
ν_{12}



$\mu_{23,12}$



$\eta_{1,31}$

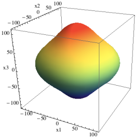


$\eta_{31,1}$

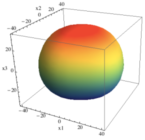
Calcium Tungstate ($CaWO_4$)

Crystal syngony: Tetragonal, $N = 7$.

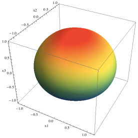
$$[C] = \begin{bmatrix} 141 & 61 & 41 & 0 & 0 & 1.9 \\ & 141 & 41 & 0 & 0 & -1.9 \\ & & 125 & 0 & 0 & 0 \\ & & & 67.4 & 0 & 0 \\ & \mathit{sym} & & & 67.4 & 0 \\ & & & & & 81.4 \end{bmatrix}$$



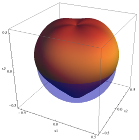
E_1



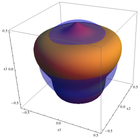
G_{12}



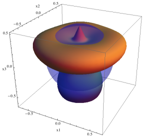
ν_{12}



$\mu_{23,12}$



$\eta_{1,31}$

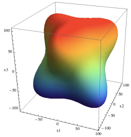


$\eta_{31,1}$

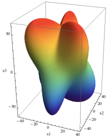
Quartz (SiO_2)

Crystal syngony: Trigonal, $N = 6$.

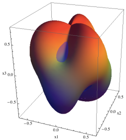
$$[C] = \begin{bmatrix} 86.8 & 7.1 & 14.4 & 24.3 & 0 & 0 \\ & 86.8 & 14.4 & -24.3 & 0 & 0 \\ & & 107.5 & 0 & 0 & 0 \\ & & & 116.4 & 0 & 0 \\ & sym & & & 116.4 & 34.4 \\ & & & & & 79.7 \end{bmatrix}$$



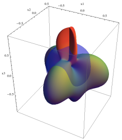
E_1



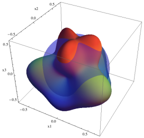
G_{12}



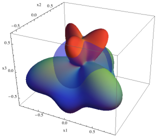
ν_{12}



$\mu_{23,12}$



$\eta_{1,31}$

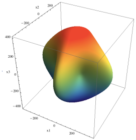


$\eta_{31,1}$

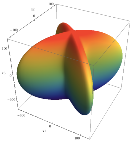
Zircon ($ZrSiO_4$)

Crystal syngony: Tetragonal, $N = 6$.

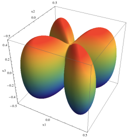
$$[C] = \begin{bmatrix} 424 & 70 & 149 & 0 & 0 & 0 \\ & 424 & 149 & 0 & 0 & 0 \\ & & 489 & 0 & 0 & 0 \\ & & & 262 & 0 & 0 \\ & sym & & & 262 & 0 \\ & & & & & 96 \end{bmatrix}$$



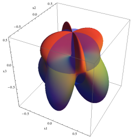
E_1



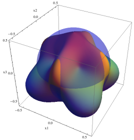
G_{12}



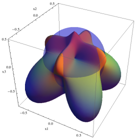
ν_{12}



$\mu_{23,12}$



$\eta_{1,31}$

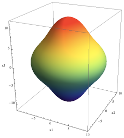


$\eta_{31,1}$

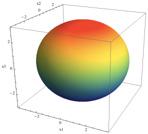
Ice (H_2O)

Crystal syngony: Hexagonal, $N = 5$.

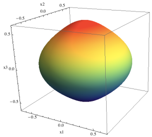
$$[C] = \begin{bmatrix} 13.5 & 6.5 & 6 & 0 & 0 & 0 \\ & 13.5 & 6 & 0 & 0 & 0 \\ & & 15 & 0 & 0 & 0 \\ & & & 6 & 0 & 0 \\ & sym & & & 6 & 0 \\ & & & & & 7 \end{bmatrix}$$



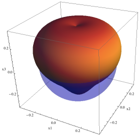
E_1



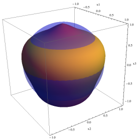
G_{12}



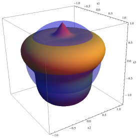
ν_{12}



$\mu_{23,12}$



$\eta_{1,31}$

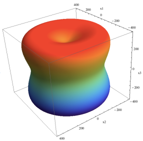


$\eta_{31,1}$

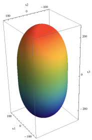
Titanium Boride (TiB_2)

Crystal syngony: Hexagonal, $N = 5$.

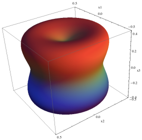
$$[C] = \begin{bmatrix} 648.3 & 404.2 & 317.7 & 0 & 0 & 0 \\ & 648.3 & 317.7 & 0 & 0 & 0 \\ & & 439.3 & 0 & 0 & 0 \\ & & & 500 & 0 & 0 \\ & sym & & & 500 & 0 \\ & & & & & 244.1 \end{bmatrix}$$



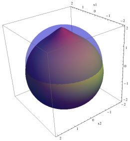
E_1



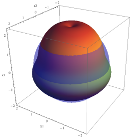
G_{12}



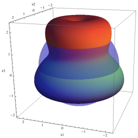
ν_{12}



$\mu_{23,12}$



$\eta_{1,31}$

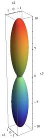


$\eta_{31,1}$

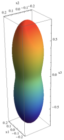
Pine Wood

Transversely isotropic, $N = 5$.

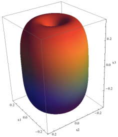
$$[C] = \begin{bmatrix} 0.45 & 0.11 & 0.13 & 0 & 0 & 0 \\ & 0.45 & 0.13 & 0 & 0 & 0 \\ & & 10.1 & 0 & 0 & 0 \\ & & & 1.5 & 0 & 0 \\ & sym & & & 1.5 & 0 \\ & & & & & 0.34 \end{bmatrix}$$



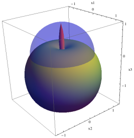
E_1



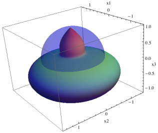
G_{12}



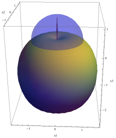
ν_{12}



$\mu_{23,12}$



$\eta_{1,31}$

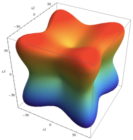


$\eta_{31,1}$

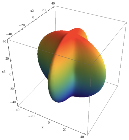
Gold (Au)

Crystal syngony: Cubic, $N = 3$.

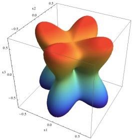
$$[C] = \begin{bmatrix} 191 & 162 & 162 & 0 & 0 & 0 \\ & 191 & 162 & 0 & 0 & 0 \\ & & 191 & 0 & 0 & 0 \\ & & & 84 & 0 & 0 \\ & sym & & & 84 & 0 \\ & & & & & 84 \end{bmatrix}$$



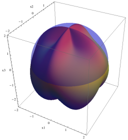
E_1



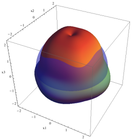
G_{12}



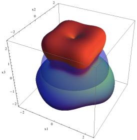
ν_{12}



$\mu_{23,12}$



$\eta_{1,31}$

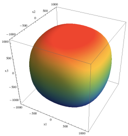


$\eta_{31,1}$

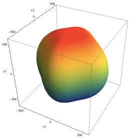
Diamond (C)

Crystal syngony: Cubic, $N = 3$.

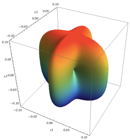
$$[C] = \begin{bmatrix} 1079 & 124 & 124 & 0 & 0 & 0 \\ & 1079 & 124 & 0 & 0 & 0 \\ & & 1079 & 0 & 0 & 0 \\ & & & 1156 & 0 & 0 \\ & \mathit{sym} & & & 1156 & 0 \\ & & & & & 1156 \end{bmatrix}$$



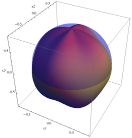
E_1



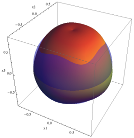
G_{12}



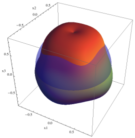
ν_{12}



$\mu_{23,12}$



$\eta_{1,31}$



$\eta_{31,1}$