UNIVERSITÀ DEGLI STUDI FIRENZE

UNIVERSITÀ DI PISA

Università degli Studi di Perugia

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International Doctorate in Civil and Environmental Engineering

Anisotropic Structures - Theory and Design

Strutture anisotrope: teoria e progetto

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Lesson 1 - April 2, 2019 - DICEA - Universitá di Firenze

Topics of the first lesson

- Generalities about anisotropy
- Anisotropic elasticity - Part 1

Generalities about anisotropy

What is anisotropy?

In ancient Greek, *άνισος* means different, unequal, and *τρόπος*, direction: anisotropy indicates hence the concept of not equivalent directions.

In some old texts, anisotropy is sometimes called æolotropy, from ancient Greek *αὶόλος*, that means changeful.

In physics, an anisotropic phenomenon is a property changing with the direction.

As a consequence, any two directions are not equivalent for such a phenomenon, but if a couple or more directions can be found whereon the phenomenon has the same behavior or value, then these directions are called equivalent.

Several physical properties can be anisotropic, because the medium where they take place or propagate is anisotropic.

This is essentially due to the structure of the solid matter itself, the most part of times organized, at a given scale, according to a geometrical scheme.

Natural examples: crystals, several natural stones (e.g. sandstone or marble), pack ice, bones, leaves or wood.

Artificial examples: laminated steel, composite materials, some structures.

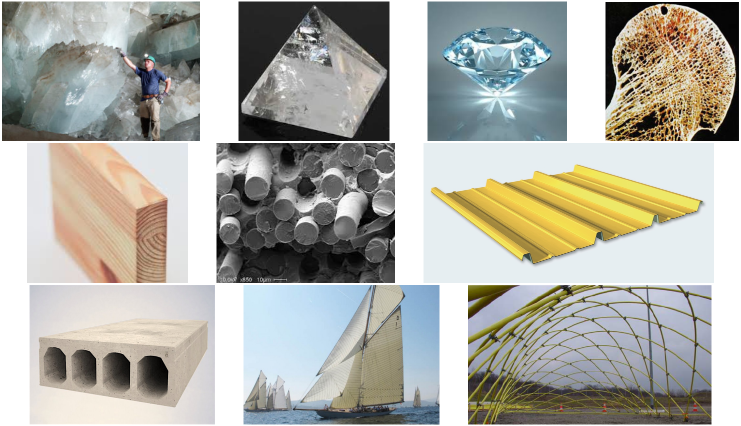

Figure: Some examples of anisotropic materials or structures.

We are interested here in the anisotropic behavior of elastic materials and structures.

Several structural materials and biological tissues, not only wood or bones, are actually anisotropic.

Different structures can be modeled, at a sufficiently large scale, as composed by a fictitious, equivalent anisotropic material.

In modern industrial applications, like aircrafts and sport devices, anisotropic composite materials reinforced by oriented fibres find an increasing use.

In civil engineering, structural solutions making use of organized schemes that induce a macroscopic anisotropic behavior (e.g. stiffened plates, spatial trusses) are more and more used, because of their capacity of bearing loads and covering large spans.

There is hence a real need of understanding how modeling and designing anisotropic structures and materials:

- on one side, for understanding the mechanical behavior of natural structures, in botanics, bio-engineering, geophysics and so on
- on the other side, for designing practical applications for different fields of the human activity

This course is an introduction to the theory and design of anisotropic elastic structures, with an insight on some advanced topics in the matter.

Mathematical consequences of anisotropy

The anisotropy of a physical property has some consequences on its mathematical description:

- the effect of the dependence upon the direction must be described → increase of the number of parameters to be used for the description of the phenomenon.
- the eventual geometrical symmetries must be taken into account, because they normally give some relations about the parameters describing the property.

Effects on algebraic operators

A physical property or phenomenon is mathematically described by an operator; this can be simply a scalar, or a vector, or more frequently a tensor of a given rank.

Often, when the property is isotropic, a simple scalar is sufficient to completely describe it, but when the same property is anisotropic, then a single scalar is no more sufficient, because the dependence upon the direction must be taken into account.

Some examples...

Thermal expansion

Strain ε produced by a change of temperature $\varDelta t$:

$$\boldsymbol{\varepsilon} = \varDelta t \boldsymbol{\alpha}; \tag{1}$$

Isotropic body: thermal expansion identical in all the directions → a single coefficient α completely describes the phenomenon ⇒

$$\boldsymbol{\alpha} = \alpha \mathbf{I} \tag{2}$$

Anisotropic body: $\boldsymbol{\alpha}$ cannot be proportional to the identity[1]:

$$\boldsymbol{\alpha} = \alpha_{ij} \mathbf{e}_i \otimes \mathbf{e}_j, \quad \alpha_{ij} = \alpha_{ji} \tag{3}$$

$\boldsymbol{\alpha}$: tensor of thermal expansion coefficients

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[1] $\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{V}$, the *dyad* $(\mathbf{a} \otimes \mathbf{b})$ is the second-rank tensor defined by the operation $(\mathbf{a} \otimes \mathbf{b})\mathbf{c} := \mathbf{b} \cdot \mathbf{c}\ \mathbf{a}$.

Ohm's law

The Ohm's law links the vectors of electric field **E** and current density **j**:

$$\mathbf{j} = \boldsymbol{\mu}\mathbf{E}; \tag{4}$$

Isotropic conductor: one parameter is sufficient to fix the law, the

conductivity μ →

$$\boldsymbol{\mu} = \mu\mathbf{I} \tag{5}$$

j is parallel and proportional to **E**.

Anisotropic conductor: $\boldsymbol{\mu}$ is not proportional to the identity and the Ohm's law depends hence upon a second-rank symmetric tensor, the conductivity tensor $\boldsymbol{\mu}$:

$$\boldsymbol{\mu} = \mu_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \quad \mu_{ij} = \mu_{ji} \tag{6}$$

j changes with the direction; it is parallel to **E** only when this is aligned with one of the principal directions of $\boldsymbol{\mu}$.

Paramagnetism and diamagnetism

Be **H** the magnetic field intensity, **L** the intensity of magnetization and **B** the magnetic induction; they are related by the law

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{L}, \tag{7}$$

$\mu_0 = 4\pi \times 10^{-7}$ H/m : magnetic permeability of vacuum.

Isotropic medium:

$$\mathbf{L} = \mu_0 \Psi \mathbf{H}, \tag{8}$$

Ψ : magnetic susceptibility of the medium, per unit of volume→

$$\mathbf{B} = \mu \mathbf{H} \quad \mu = \mu_0 (1 + \Psi), \tag{9}$$

μ: magnetic permeability of the medium. **H**, **L** and **B** are parallel.

Anisotropic medium:

$$\mathbf{L} = \mu_0 \boldsymbol{\Psi} \mathbf{H}, \quad \mathbf{B} = \boldsymbol{\mu} \mathbf{H}. \tag{10}$$

$\boldsymbol{\Psi}$: magnetic susceptibility tensor

$\boldsymbol{\mu} = \mu_0(\mathbf{I} + \boldsymbol{\Psi})$: permeability tensor of the substance.

H, **L** and **B** are all parallel $\iff$ **H** is parallel to one of the eigenvectors (principal axes) of $\boldsymbol{\Psi}$.

A crystal is said to be paramagnetic along the principal directions $\mathbf{e}_i$ of $\boldsymbol{\Psi}$ if the corresponding eigenvalue $\Psi_i > 0$, diamagnetic if $\Psi_i < 0$.

Thermal conductivity

Heat flux vector **h**:

$$\mathbf{h} = -\mathbf{k}\nabla t, \tag{11}$$

k : thermal conductivity tensor.

Isotropic body:

$$\mathbf{k} = k\mathbf{I} \tag{12}$$

k: thermal conductivity of the medium; **h** is parallel to ∇t

Anisotropic body:

$$\mathbf{k} = k_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \quad k_{ij} = k_{ji} \tag{13}$$

h and ∇t are parallel $\iff$ ∇t is parallel to one of the eigenvectors of **k**

Piezoelectricity

Some crystals, thanks to their particular type of anisotropy, develop an electric polarization if stressed by applied forces (direct piezoelectric effect, from ancient Greek $\pi\iota\acute{\varepsilon}\zeta\varepsilon\iota\nu$ that means to press):

$$\mathbf{P} = \mathcal{D}\boldsymbol{\sigma}, \tag{14}$$

$\mathcal{D}$: third-rank tensor of piezoelectric moduli.

By components:

$$P_i = D_{ijk}\sigma_{jk}. \tag{15}$$

$\mathcal{D}$ describes also the converse piezoelectric effect or Lippmann's effect, that connects the strain to the electric field:

$$\boldsymbol{\varepsilon} = \mathcal{D}\mathbf{E}\,: \tag{16}$$

the electric current strains a piezoelectric crystal.

By components:

$$\varepsilon_{jk} = D_{ijk} E_i. \tag{17}$$

The 27 components of $\mathcal{D}$ are not all independent:

$$D_{ijk} = D_{ikj}, \tag{18}$$

→ 18 independent piezoelectric moduli in the most general case; their number can be further reduced taking into account for the specific symmetries of a crystal.

The Hooke's law

The Hooke's law is the linear relation between the stress, $\boldsymbol{\sigma}$, and strain, $\boldsymbol{\varepsilon}$, in an elastic body

$$\boldsymbol{\sigma} = \mathbb{E}\boldsymbol{\varepsilon} \tag{19}$$

$\mathbb{E}$ is a fourth-rank tensor

Isotropic body: $\mathbb{E}$ is completely described by two invariant parameters

Anisotropic body: 21 frame-dependent parameters are needed

This is just the subject of this course!

A general consideration about anisotropic phenomena

From above it appears that anisotropy is an intrinsic quality of a continuum, originated by the existence, or absence, of some symmetries in the geometric distribution of the matter.

This is indeed a classical point of view, but it is important to understand that the anisotropy of the continuum is just a necessary, but not sufficient, condition for a given physical property to be anisotropic on such a continuum.

Two factors, together, determine whether a property is or not anisotropic: the intrinsic anisotropy of the continuum, i.e. its true material symmetries, and the type of algebraic operator describing a given physical property.

For the same continuum, it is possible that some physical properties described by high order tensorial operators are anisotropic, while some other ones, described by low order tensors, are perfectly isotropic.

A classical example: cubic symmetry in 3D or square symmetry in planar media.

For such a material, the elastic behavior, described by a fourth-rank tensor, is not isotropic, contrarily to what is often believed, but the thermoelastic behavior, described by the second-rank tensor of thermal expansion coefficients, is isotropic, as well as any other physical property depending upon a second-rank tensor.

This fact has been condensed, in some way, in the empirical Principle of Neumann, see further.

Finally, it is much more correct to talk about the anisotropy of a physical property on a continuum rather than that of anisotropy of the continuum itself.

Some few words about geometrical symmetries

A body is said to have a symmetry when it can be brought to coincidence with itself by a transformation, called a covering operation, that moves any of his points. In such a case, we say that the body allows the transformation.

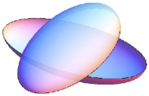
Possible covering operations:

- rotation about an angle through a definite, or even indefinite, angle; in this case the body possesses an axis of symmetry;
- reflexion in a plane; the body has then a plane of symmetry;
- a combination of rotations and reflexions.

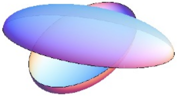
Particularly important combination: a rotation combined with a reflexion in the plane perpendicular to the symmetry axis → ∃ axis of alternating symmetry.

Special case: central perversion→ the angle of rotation is π; ∃ centre of symmetry

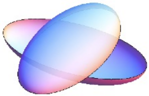
(a) Ellipsoid: the three principal planes are planes of symmetry and the three principal axes are axes of symmetry.



(b) The splitted ellipsoid: the upper part is rotated through $\pi/2$ about the vertical axis.



(c) Rotation through $\pi/2$ about the vertical axis (or reflexion in the horizontal plane).



(d) Reflexion in the horizontal plane (or rotation through $\pi/2$ about the vertical axis): the original figure (b) is recovered.

Some basic elements about crystals

A crystal is a typical anisotropic body: the geometrical disposition of the elementary constituents, atoms or molecules, on a lattice induces the anisotropy of the physical phenomena that take place in the crystal.

A lattice is an array of equally spaced and similarly situated points; in the physical reality of the crystals, these points are occupied by atoms or molecules. The points are said to be similarly situated when the rest of the lattice appears the same, and in the same orientation, when it is viewed from them.

One can consider three sets of evenly spaced parallel planes that pass through the lattice points and divide up the crystal into identical elementary parallelepipeds, the primitive unit cells, that have lattice points only at the corners.

Nevertheless, it is often better to chose a larger unit cell, having lattice points not only in the corners, but for instance at the centre of the faces; such celles are called multiple primitive unit cells.

In all the cases, a cell is defined by the lengths and directions of three vectors, denoted in the following by **a**, **b** and **c**, corresponding to three non parallel of its edges,

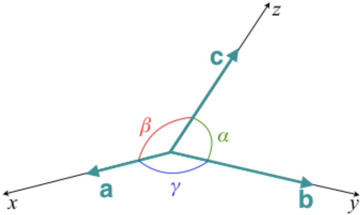


Figure: Geometrical elements of a unit cell.

Bravais lattice: infinite lattice of points generated by a set of discretized translations ruled by a law of the type

$$\mathbf{u} = p\mathbf{a} + q\mathbf{b} + r\mathbf{c}, \tag{20}$$

with p, q and r integers.

The lattice is independent from the choice of the position of the vector $\mathbf{u}$, so it appears the same when viewed from any lattice point.

It can be shown that there are only 14 possible Bravais lattices in 3D. They constitute the only possibilities for filling the space through the repetition of an elementary motive→ the unit cell of the Bravais lattice, called also the Bravais cell.

The 14 Bravais cells are usually grouped into seven distinct crystal systems: *Triclinic*, *Monoclinic*, *Orthorhombic*, *Tetragonal*, *Trigonal*, *Hexagonal* and *Cubic*.

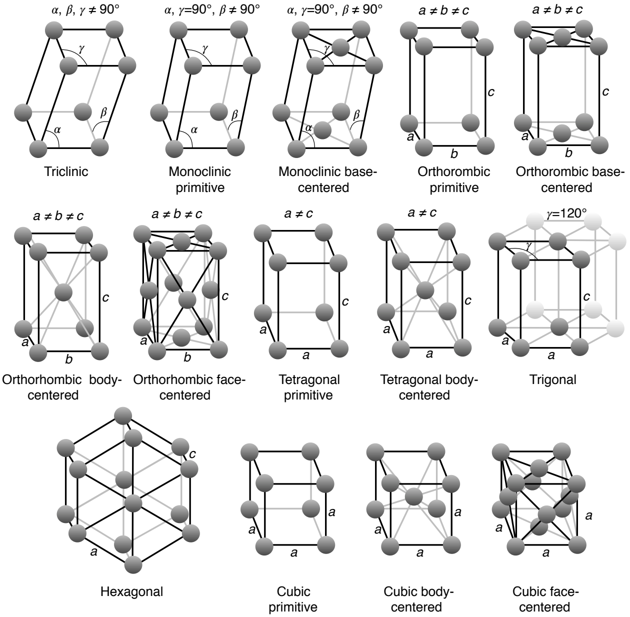


Figure: The 14 Bravais cells

A crystal system is sometimes called a syngony

Authors do not agree completely on their number (some authors group together the hexagonal and trigonal systems, so listing only 6 crystal syngonies, like Love who follows the classification given by Lewis), nor on their names (e.g. Lewis names Rhombic or Prismatic the Orthorombic syngony, and Rhombohedral the Trigonal system, included in the Hexagonal syngony).

The physical properties of crystals depend upon the symmetry elements of the crystals and on their combinations.

The only possible symmetry elements for a crystal are (combinations are possible):

- a centre of symmetry;
- a plane of symmetry;
- a 2-, 3-, 4- or 6-fold axis of symmetry;
- a 2-, 3-, 4- or 6-fold axis of alternating symmetry.

A $n-$fold axis of symmetry allows a covering that is a rotation through an angle of $2\pi/n \Rightarrow \pi, 2\pi/3, \pi/2$ and $\pi/3$ are the only rotation coverings in a crystal (the same is true also for an axis of alternating symmetry).

It can be shown that the number of groups of covering operations for crystals are 32, defining the 32 classes of crystals.

Once the class of a crystal known, i.e. all its symmetries, one can determine the form of the strain energy function, and hence the number and type of independent elastic constants.

There are only 10 possible cases, the elastic syngonies, to differentiate them from the crystal syngonies, whose number is limited to 7 (or to 6 for some authors).

Though the elastic syngonies are 10, the number of independent anisotropic elastic constants can get only 8 distinct values: 21, 13, 9, 7, 6, 5, 3 and 2; the last case is that of isotropy, impossible for crystals.

Classification of crystals

There are different possible classifications for crystals, depending upon the elements of symmetry; the different classifications use different nomenclatures and make a different use of the symmetry elements. The most well known are :

- the Hermann-Mauguin classification with a notation based upon the essential symmetry elements of a class, adopted as the standard one since 1935 by the International Tables for Crystallography
- the classification of Schoenflies, used in spectroscopy
- the Lekhnitskii classification uses a different nomenclature for the crystal classes, based upon a complete listing of the symmetry elements
- Voigt has given a number to each one of the 32 classes
- Lewis and Miers have named them, of course with different names

			Classification			
Syngony	Symmetry elements	Examples	Voigt	Sch	H-M	N
Triclinic	none	$H_3BO_3, K_2Cr_2O_7, CuSO_4 \cdot 5H_2O$	1	S_2	$\bar{1}$	21
			2	C_1	1	21
Monoclinic	a 2-fold a-s or a-a-s	$Na_2SO_4 \cdot 10H_2O$	3	C_2^h	$2/m$	13
			4	S	m	13
			5	C_2	2	13
Orthorhombic	3 mutually orthogonal 2-fold a-s or a-a-s	$KNO_3, BaSO_4$	6	V_h	mmm	9
			7	V	222	9
			8	C_2^v	$mm2$	9
Trigonal	a 3-fold a-s or a-a-s	$HgS, CaCO_3$	9	S_6^u	$\bar{3}m$	6
			10	D_3	32	6
			11	C_3^v	$3m$	6
			12	S_6	$\bar{3}$	7
			13	C_3	3	7
Tetragonal	a 4-fold a-s or a-a-s	$CaSO_4, TiO_2$	14	D_4^h	$4/mmm$	6
			15	D_4	4222	6
			16	C_4^v	$4mm$	6
			17	C_4^h	$4/m$	7
			18	C_4	4	7
			19	S_4^u	$\bar{4}2m$	6
			20	S_4	$\bar{4}$	7
Hexagonal	a 6-fold a-s or a-a-s	CdS, ZnO, graphite	21	D_6^h	$6/mmm$	5
			22	C_6^v	$6mm$	5
			23	D_6	622	5
			24	C_6^h	$6/m$	5
			25	C_6	6	5
			26	D_3^h	$\bar{6}m2$	5
			27	C_3^h	$\bar{6}$	5
Cubic	four 3-fold a-s arranged as the cubic diagonal	$NaCl, Cu$, Zinc Blende	28	O^h	$m\bar{3}m$	3
			29	O	432	3
			30	T^d	$\bar{4}3m$	3
			31	T^h	$m\bar{3}$	3
			32	T	332	3

Note: Sch= Schoenflies, H-M= Hermann-Mauguin; a-s= axis of symmetry, a-a-s: axis of alternate symmetry. N: number of independent elastic constants.

The Neumann's Principle

The Neumann's Principle is an empirical principle relating the symmetry class of a crystal and the symmetries of its anisotropic properties:

> Any kind of symmetry, possessed by the crystallographic form of a material, is possessed also by the material in respect of every physical property.

This principle does not establish a one to one correspondence between the crystal symmetries and the physical symmetries, but merely the inclusion of all the symmetries of the crystal in the set of physical symmetries $\Rightarrow$ some physical properties can have symmetries not possessed by the crystal (e.g. cubic crystals are optically isotropic; isotropy includes all the cubic symmetries)

Some known ingredients of elasticity

We recall here some fundamental equations, that will be occasionally used in the following of this text.

- Infinitesimal strain tensor: for a displacement vector field $\mathbf{u}(x_1, x_2, x_3)$ such that $|u_{i,j}| \ll 1 \; \forall i, j = 1, 2, 3, \; u_{i,j} = \frac{\partial u_i}{\partial x_j}$

  $$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^\top) = \varepsilon_{ij} \mathbf{e}_i \otimes \mathbf{e}_j, \; i, j = 1, 2, 3, \tag{21}$$

  already introduced, is the measure of strain;

  $$\varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2} \quad i, j = 1, 2, 3. \tag{22}$$

  $\nabla \mathbf{u}$ is the displacement gradient:

  $$\nabla \mathbf{u} = u_{i,j} \mathbf{e}_i \otimes \mathbf{e}_j, \; i, j = 1, 2, 3, \tag{23}$$

- Equation of motion:

$$\mathrm{div}\boldsymbol{\sigma} = \mathbf{b} \;\rightarrow\; \sigma_{ij,j} = b_i \;\forall i, j = 1, 2, 3, \rightarrow \tag{24}$$

$$\begin{aligned} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} &= b_1, \\ \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} &= b_2, \\ \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} &= b_3, \end{aligned} \tag{25}$$

where **b** is the body vector, that includes the mass acceleration and the reversed body force **f**:

$$\mathbf{b} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathbf{f}. \tag{26}$$

If $\frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathbf{o}$ everywhere → equilibrium equation.

- The Saint Venant-Beltrami compatibility equations give the integrability necessary conditions for the components of ε; in compact form they are:

$$\varepsilon_{ij,kl}+\varepsilon_{kl,ij}-\varepsilon_{ik,jl}-\varepsilon_{jl,ik}=0; \tag{27}$$

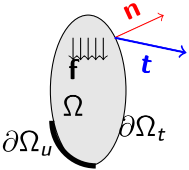
these are 81 equations, but only 6 are not identities, as it can be checked with some work but without difficulty:

$$\begin{aligned}
2\varepsilon_{12,12}&=\varepsilon_{11,22}+\varepsilon_{22,11},\\
2\varepsilon_{23,23}&=\varepsilon_{22,33}+\varepsilon_{33,22},\\
2\varepsilon_{31,31}&=\varepsilon_{11,33}+\varepsilon_{33,11},\\
\varepsilon_{11,23}+\varepsilon_{23,11}&=\varepsilon_{13,12}+\varepsilon_{12,13},\\
\varepsilon_{22,13}+\varepsilon_{13,22}&=\varepsilon_{12,23}+\varepsilon_{23,12},\\
\varepsilon_{33,12}+\varepsilon_{12,33}&=\varepsilon_{23,13}+\varepsilon_{13,23}.
\end{aligned} \tag{28}$$

Anisotropic elasticity

Part 1

The Hooke's law for anisotropic bodies



Be Ω a body acted upon by body forces $\mathbf{f}$ and by surface tractions $\mathbf{t}$ on its frontier $\partial\Omega$ whose outward unit normal is $\mathbf{n}$.

We consider a small arbitrary variation $\delta\mathbf{u}$ of the displacement field on Ω, compatible with the given boundary conditions and satisfying the kinematical conditions

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^\top), \qquad (29)$$

The total mechanical work dW done by the applied forces can be transformed as follows

$$\begin{aligned} dW = \int_\Omega \mathbf{f}\cdot\delta\mathbf{u}\; d\omega + \int_{\partial\Omega} \mathbf{t}\cdot\delta\mathbf{u}\; ds = \int_\Omega \mathbf{f}\cdot\delta\mathbf{u}\; d\omega + \int_{\partial\Omega} \boldsymbol{\sigma}\mathbf{n}\cdot\delta\mathbf{u}\; ds = \\ \int_\Omega \mathbf{f}\cdot\delta\mathbf{u}\; d\omega + \int_{\partial\Omega} \boldsymbol{\sigma}\delta\mathbf{u}\cdot\mathbf{n}\; ds = \int_\Omega [\mathbf{f}\cdot\delta\mathbf{u} + \mathrm{div}(\boldsymbol{\sigma}\delta\mathbf{u})]\; d\omega = \\ \int_\Omega [(\mathbf{f} + \mathrm{div}\boldsymbol{\sigma})\cdot\delta\mathbf{u} + \boldsymbol{\sigma}\cdot\nabla\delta\mathbf{u}]\; d\omega = \int_\Omega \boldsymbol{\sigma}\cdot\delta\boldsymbol{\varepsilon}\; d\omega \end{aligned} \tag{30}$$

In establishing eq. (30) we have used successively the Cauchy's stress theorem, the symmetry of $\boldsymbol{\sigma}$, the Gauss theorem, two standard results for tensor and vector fields[2] and the motion equation.

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[2] Namely, we have used the identity $\mathrm{div}(\mathbf{L}^\top\mathbf{v}) = \mathrm{div}\mathbf{L}\cdot\mathbf{v} + \mathbf{L}\cdot\nabla\mathbf{v}$, and the fact that $\mathbf{L}\cdot\frac{\nabla\mathbf{v}+\nabla\mathbf{v}^\top}{2} = \mathbf{L}\cdot\nabla\mathbf{v} \;\; \forall\mathbf{L} = \mathbf{L}^\top$

The quantity

$$\delta V = \boldsymbol{\sigma} \cdot \delta \boldsymbol{\varepsilon} = \sigma_{ij} \delta \varepsilon_{ij}, \tag{31}$$

represents the variation of the internal energy of the body per unit of volume produced by a small variation of the strain state.

Following the energetic approach of Green, we define as elastic a body for which the total variation ΔV of the internal energy due to a finite transformation from a state A to a state B is independent from the integration path.

In particular

$$\Delta V = \int_A^B \delta V = V_B - V_A \tag{32}$$

must then be null for any transformation where A=B.

The strain energy density

Hence, for an elastic body δV must be the exact differential of a scalar function $V(\varepsilon)$, the strain energy density or elastic potential such that

$$dV = \boldsymbol{\sigma} \cdot d\boldsymbol{\varepsilon} = \sigma_{ij} d\varepsilon_{ij}, \tag{33}$$

which gives the Green's formula

$$\sigma_{ij} = \frac{\partial V}{\partial \varepsilon_{ij}}. \tag{34}$$

We now postulate that in the initial state, i.e. when the body is not acted upon by forces, $\varepsilon = \mathbf{o}$ and $\boldsymbol{\sigma} = \mathbf{o}$, i.e. the body is unstrained and unstressed in its natural state. Then, developing $V(\varepsilon)$ in a Taylor series about $\boldsymbol{\varepsilon} = \mathbf{o}$ we get

$$V(\boldsymbol{\varepsilon}) = V(\boldsymbol{\varepsilon} = \mathbf{o}) + \left.\frac{\partial V}{\partial \varepsilon_{ij}}\right|_{\boldsymbol{\varepsilon}=\mathbf{o}} \varepsilon_{ij} + \frac{1}{2} \left.\frac{\partial^2 V}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}\right|_{\boldsymbol{\varepsilon}=\mathbf{o}} \varepsilon_{ij}\varepsilon_{kl} + \ldots \tag{35}$$

The elastic tensor

After putting $V(\boldsymbol{\varepsilon}=\mathbf{o})=0$ and limiting the development to the first non null term, which is correct for small strains, gives

$$V=\frac{1}{2}\left.\frac{\partial^2 V}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}}\right|_{\boldsymbol{\varepsilon}=\mathbf{o}}\varepsilon_{ij}\varepsilon_{kl};\tag{36}$$

the second derivatives in the above equation are linear operators depending upon four indexes; they are the components of a fourth-rank tensor $\mathbb{E}$,

$$E_{ijkl}:=\left.\frac{\partial^2 V}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}}\right|_{\boldsymbol{\varepsilon}=\mathbf{o}},\tag{37}$$

the (stiffness) elasticity tensor, so that

$$V=\frac{1}{2}E_{ijkl}\varepsilon_{ij}\varepsilon_{kl}=\frac{1}{2}\boldsymbol{\varepsilon}\cdot\mathbb{E}\boldsymbol{\varepsilon}.\tag{38}$$

Major symmetries

$\mathbb{E}$ describes the elastic response of the continuum; its 81 Cartesian components are the elastic moduli E_{ijkl}

The independent elastic moduli are far less than 81: by the Schwarz theorem we get

$$E_{ijkl} = \frac{\partial^2 V}{\partial \varepsilon_{kl} \partial \varepsilon_{ij}} = \frac{\partial^2 V}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = E_{klij}; \tag{39}$$

the above 15 relations are known as major symmetries and reduce the number of independent moduli of $\mathbb{E}$ from 81 to 66.

The Hooke's law

Applying the Green's formula (34) we get

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl} \;\rightarrow\; \boldsymbol{\sigma} = \mathbb{E}\boldsymbol{\varepsilon}. \tag{40}$$

This is the Hooke's law (1660), establishing a linear relation between stress and strain.

This linearity is a direct consequence of the quadratic structure of V and of the Green's formula.

Though initially formulated for isotropic bodies, it is the basic law of elasticity also for the more general case of anisotropic continua.

Minor symmetries

Through the Hooke's law, because of the symmetry of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$,

$$\begin{aligned} &\sigma_{ij} = \sigma_{ji} \quad \text{and} \quad \varepsilon_{ij} = \varepsilon_{ji} \quad \Rightarrow \\ &E_{ijkl} = E_{jikl} = E_{ijlk} = E_{jilk} \quad \forall i, j, k, l \in \{1, 2, 3\}. \end{aligned} \tag{41}$$

The above 45 relations are the minor symmetries and reduce the independent moduli to only 21.

This is the highest number of independent moduli that an elastic material can have. In such a case, the material is completely anisotropic or triclinic.

Another approach

An alternative, classical, approach to elasticity is to postulate the Hooke's law and the existence of V; once obtained the Green's formula, using the Schwarz theorem gives again the major symmetries, while the minor ones are still given by the symmetry of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$.

Then, the expression of V is obtained integrating dV:

$$\begin{gathered} dV = \boldsymbol{\sigma} \cdot d\boldsymbol{\varepsilon} = \mathbb{E}\boldsymbol{\varepsilon} \cdot d\boldsymbol{\varepsilon} = E_{ijkl}\varepsilon_{kl}d\varepsilon_{ij} \ \rightarrow \\ V = \frac{1}{2}E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} = \frac{1}{2}\boldsymbol{\varepsilon} \cdot \mathbb{E}\boldsymbol{\varepsilon}. \end{gathered} \tag{42}$$

Material dependent reductions

The above two reductions of the number of independent components of $\mathbb{E}$ are the only universal ones, i.e. valid for any elastic classical material (the so-called Green's materials).

Further reductions of the number of independent moduli can be obtained only if special conditions, not universal but depending upon the material type, are introduced.

Such conditions are called elastic or material symmetries, and indicate the invariance of some elastic moduli under some geometric transformations.

We examine hence the effects on $\mathbb{E}$ of elastic symmetries in the following.

The stress energy

Injecting the Hooke's law into the expression (38) of V we get also

$$V = \frac{1}{2}\boldsymbol{\sigma} \cdot \varepsilon. \tag{43}$$

Let us now consider the inverse of the Hooke's law:

$$\varepsilon = \mathbb{Z}\sigma, \quad \mathbb{Z} = \mathbb{E}^{-1}, \tag{44}$$

with $\mathbb{Z}$ the compliance elasticity tensor; introducing this last equation for ε into (38) gives

$$V = \frac{1}{2}\boldsymbol{\sigma} \cdot \mathbb{Z}\boldsymbol{\sigma}, \tag{45}$$

an expression called stress energy density in the literature.

A warning!

A remark: in this section, the word symmetry has been used for denoting the equivalence of the positions of an index for two or more components of the elasticity tensor.

To make the distinction with the concept of elastic symmetry, the expression tensor or index symmetry could be used.

Anyway, the reader should be aware of the fact that the same word, symmetry, can have two rather different meanings in our context.

Before examining the elastic symmetries, we need to recall some *ingredients* and introduce two notations.

The Voigt's notation

The general, tensorial, expression of the Hooke's law needs the use of quantities with four indexes,

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl}, \tag{46}$$

which can be somewhat cumbersome and heavy.

That is why some simplified notations have been proposed. In particular, they allow for a matrix representation of (46); these formalisms switch the algebra from that of a 4^{th}-rank tensor to that of a 6x6 square symmetric matrix.

In the formalism of Voigt (1910), the stress and strain tensors are written as :

$$\{\sigma\} = \left\{ \begin{array}{c} \sigma_1 = \sigma_{11} \\ \sigma_2 = \sigma_{22} \\ \sigma_3 = \sigma_{33} \\ \sigma_4 = \sigma_{23} \\ \sigma_5 = \sigma_{31} \\ \sigma_6 = \sigma_{12} \end{array} \right\}, \quad \{\varepsilon\} = \left\{ \begin{array}{c} \varepsilon_1 = \varepsilon_{11} \\ \varepsilon_2 = \varepsilon_{22} \\ \varepsilon_3 = \varepsilon_{33} \\ \varepsilon_4 = 2\varepsilon_{23} \\ \varepsilon_5 = 2\varepsilon_{31} \\ \varepsilon_6 = 2\varepsilon_{12} \end{array} \right\}. \qquad (47)$$

Eq. (47) shows the relations and order established for the indexes by the Voigt's notation:

$$11 \rightarrow 1, \quad 22 \rightarrow 2, \quad 33 \rightarrow 3, \quad 23 \rightarrow 4, \quad 31 \rightarrow 5, \quad 12 \rightarrow 6. \qquad (48)$$

The introduction of the coefficient 2 for the terms $\varepsilon_4, \varepsilon_5$ and ε_6 is needed for taking into account for the symmetry of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ in the Hooke's law.

This fact imposes some accuracy in the use of the Voigt's notation, because the algebras for tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are not completely the same, namely for their transformation upon axes rotation and tensor inversion.

The Voigt's notation transforms hence second rank symmetric tensors into column vectors; correspondingly, the fourth-rank elasticity tensor is transformed into a 6x6 symmetric square matrix, the symmetry of such a matrix corresponding to the major symmetries of $\mathbb{E}$.

According to the index transformation rule (48), the matrix form of the Hooke's law with the Voigt's notation is

$$\{\sigma\} = [C]\{\varepsilon\} \quad \rightarrow \tag{49}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}. \tag{50}$$

The name $[C]$ is usually preferred to $\mathbb{E}$ to make a clear distinction between the tensor and matrix representation.

There is a perfect coincidence between the E_{ijkl} and the C_{pq}, thanks to rule (48), e.g. $E_{2312} = C_{46}, E_{1322} = C_{52}$ etc.

Let us now consider the inverse of the Hooke's law:

$$\varepsilon = \mathbb{Z}\sigma, \quad \mathbb{Z} = \mathbb{E}^{-1}, \tag{51}$$

that in the Voigt's notation is

$$\{\varepsilon\} = [S]\,\{\sigma\} \quad \rightarrow \tag{52}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}. \tag{53}$$

A consequence of the introduction of the 2s in $(47)_2$ is that, unlike the case of $[C]$, not all the components of $[S]$ are equal to the corresponding ones of $\mathbb{Z}$:

$$[S_{ij}] = \left[\begin{array}{c|c} Z_{ppqq} & 2Z_{pprs} \\ \hline 2Z_{pprs} & 4Z_{pqrs} \end{array}\right] \quad \rightarrow$$

$$\left[\begin{array}{ccc|ccc} S_{11} = Z_{1111} & S_{12} = Z_{1122} & S_{13} = Z_{1133} & S_{14} = 2Z_{1123} & S_{15} = 2Z_{1131} & S_{16} = 2Z_{1112} \\ & S_{22} = Z_{2222} & S_{23} = Z_{2233} & S_{24} = 2Z_{2223} & S_{25} = 2Z_{2231} & S_{26} = 2Z_{2212} \\ & & S_{33} = Z_{3333} & S_{34} = 2Z_{3323} & S_{35} = 2Z_{3331} & S_{36} = 2Z_{3312} \\ \hline & & & S_{44} = 4Z_{2323} & S_{45} = 4Z_{2331} & S_{46} = 4Z_{2312} \\ & sym & & & S_{55} = 4Z_{3131} & S_{56} = 4Z_{3112} \\ & & & & & S_{66} = 4Z_{1212} \end{array}\right]. \tag{54}$$

The Voigt's notation implies a different algebra for $\mathbb{E}$ and $\mathbb{S}$: $[C]$ and $[S]$ do not represent some 2^{nd}-rank tensors in $\mathbb{R}^6$.

Practically, the Voigt's notation needs some carefulness, not only in the differences between $[C]$ and $[S]$, but also in the transformations produced by axes rotation.

The Kelvin's notation

The Kelvin's notation (by somebody named Mandel's notation) was proposed by Lord Kelvin as early as 1856, but, probably because making use of irrational quantities, it has not known an as large success as the Voigt's notation.

Nevertheless, rather recently a new attention has been brought by scientists on it, mainly for its algebraic properties: the Kelvin's notation has not the drawbacks of the Voigt's one, as it will be seen below.

The Kelvin's notation is different from the Voigt's one in that the coefficients 2 affecting ε, eq. (47), are equally distributed over $\boldsymbol{\sigma}$ and ε, in such a way their product still amounts to 2:

$$\{\sigma\} = \left\{ \begin{array}{c} \sigma_1 = \sigma_{11} \\ \sigma_2 = \sigma_{22} \\ \sigma_3 = \sigma_{33} \\ \sigma_4 = \sqrt{2}\sigma_{23} \\ \sigma_5 = \sqrt{2}\sigma_{31} \\ \sigma_6 = \sqrt{2}\sigma_{12} \end{array} \right\}, \quad \{\varepsilon\} = \left\{ \begin{array}{c} \varepsilon_1 = \varepsilon_{11} \\ \varepsilon_2 = \varepsilon_{22} \\ \varepsilon_3 = \varepsilon_{33} \\ \varepsilon_4 = \sqrt{2}\varepsilon_{23} \\ \varepsilon_5 = \sqrt{2}\varepsilon_{31} \\ \varepsilon_6 = \sqrt{2}\varepsilon_{12} \end{array} \right\}. \qquad (55)$$

In this way, eqs. (50) and (53) still hold but there is no difference between $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ in the transition from the tensor to the matrix representation.

In particular, the way the components of $[C]$ and $[S]$ are deduced from the corresponding ones of $\mathbb{E}$ and $\mathbb{Z}$, are exactly the same (no summation over dummy indexes):

$$[C_{ij}] = \left[\begin{array}{c|c} E_{ppqq} & \sqrt{2}E_{pprs} \\ \hline \sqrt{2}E_{pprs} & 2E_{pqrs} \end{array}\right], \quad [S_{ij}] = \left[\begin{array}{c|c} Z_{ppqq} & \sqrt{2}Z_{pprs} \\ \hline \sqrt{2}Z_{pprs} & 2Z_{pqrs} \end{array}\right]. \tag{56}$$

The above symbolic relations can be easily translated in the detailed expressions of C_{ij} and S_{ij}, applying a scheme quite similar to that detailed in eq. (54).

Merhabadi and Cowin (1990), have shown that the Kelvin's notation gives a representation of elasticity by matrices, $[C]$ and $[S]$, representing 2^{nd}-rank symmetric tensors in $\mathbb{R}^6$, which is not the case with the Voigt's notation.

Hence, the Kelvin's notation transfers the algebra of elasticity from 4^{th}-rank tensors in $\mathbb{R}^3$, to 2^{nd}-rank tensors in $\mathbb{R}^6$.

This fact has some advantages, for instance the rotation of matrices $[C]$ and $[S]$ is made in the same way, unlike with the Voigt's notation.

Mechanical meaning of the anisotropic elastic components

In the most general case of a triclinic material, the number of independent elastic moduli is as great as 21.

It is important to understand the mechanical meaning of these parameters, because, unlike in the case of an isotropic material, some unusual, strange mechanical effects can arise in anisotropy.

To discover these effects and connect them to particular elastic parameters, it is worth to use the compliances, i.e. the components of $[S]$ or $\mathbb{Z}$.

Let us consider a cube of a triclinic material, subjected to the only traction σ_1. In such a case, eq. (53) gives

$$\begin{aligned} \varepsilon_k &= S_{k1}\sigma_1 \quad \forall k = 1, ..., 6, \quad \rightarrow \\ \varepsilon_{ij} &= Z_{ij11}\sigma_{11} \quad \forall i, j = 1, 2, 3. \end{aligned} \tag{57}$$

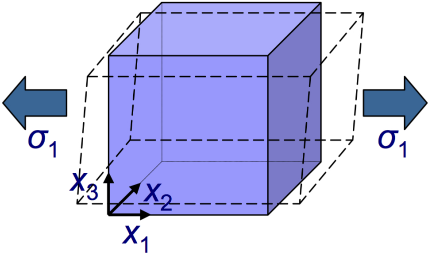


Figure: Anisotropic stretched cube.

$\rightarrow$ In a completely anisotropic body all the components of ε are not null: a normal stress produces also shearing strains $\rightarrow$ The coupling effects are not restricted to the only Poisson's effect, due to the terms $S_{ij},\ i,j=1,2,3,\ i\neq j$: in the anisotropic case, there is also a coupling between normal stress and shear strain, due to terms $S_{kl},\ k=4,5,6,\ l=1,2,3$.

In addition, generally speaking $S_{12}\neq S_{23}\neq S_{31}\rightarrow$ the Poisson's effect is different in the orthogonal directions, i.e. $\varepsilon_2\neq\varepsilon_3$.

In the same way, usually $S_{4k}\neq S_{5k}\neq S_{6k},\ k=1,2,3$, so that also for the shearing strains it is $\varepsilon_4\neq\varepsilon_5\neq\varepsilon_6$.

Finally, the anisotropic cube does not only change its volume under the unique action of a traction, like in isotropic bodies, but it changes also its form: it becomes a prism with no orthogonal faces.

Let us now consider the same cube submitted to a unique shear stress, say σ_5; eq. (53) gives then

$$\begin{aligned}
\varepsilon_k &= S_{k5}\sigma_5 \;\; \forall k = 1, ..., 6, \;\; \rightarrow \\
\varepsilon_{ij} &= Z_{ij31}\sigma_{31} \;\; \forall i, j = 1, 2, 3.
\end{aligned} \tag{58}$$

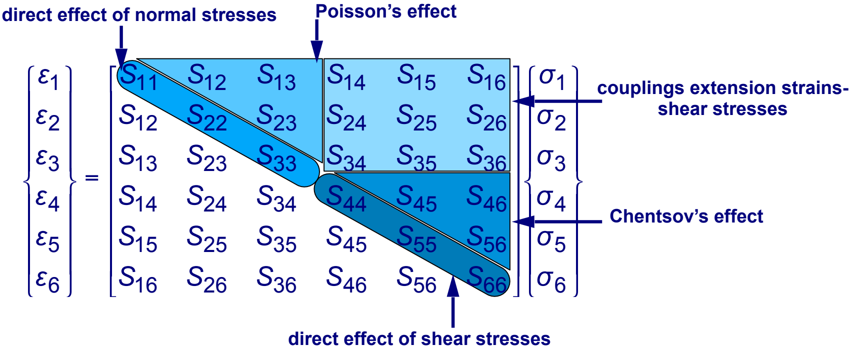
This shows the existence of a coupling between shear stresses and extension strains, due to the terms $S_{lk},\ k = 4, 5, 6,\ l = 1, 2, 3$ and also a coupling between a shear stress and the shearing strains in orthogonal planes, due to the terms $S_{ij},\ i, j = 4, 5, 6,\ i \neq j$.

This last effect is called the Chentsov's effect: it is analogous to the Poisson's effect, but it concerns shear stresses and strains.

Also in this case, these couplings are not necessarily the same in all the planes, because generally speaking $S_{l4} \neq S_{l5} \neq S_{l6},\ l = 1, 2, 3$ and $S_{45} \neq S_{56} \neq S_{64}$.

Submitted to simple shear stress, the cube changes not only its shape, but also its volume, unlike in the case of isotropic bodies.

Finally, $[S]$, and also $[C]$, can be subdivided into parts in charge of a particular effect:



direct effect of normal stresses

Poisson's effect

couplings extension strains-shear stresses

Chentsov's effect

direct effect of shear stresses

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

Figure: Partition of the compliance matrix by mechanical effects.

Taking into account for the elastic symmetries

We now go back to the third reduction of the E_{ijkl}; as said, this can be done upon introduction of the elastic symmetries.

When some kind of symmetry in the behavior is present, then some equivalent directions exist, whereon the behavior is the same.

Just because the behavior is the same along equivalent directions, the forms of the elasticity matrix $[C]$ and of the strain energy are the same in two frames related by a covering operation.

This gives some relations among the components of $[C]$, e.g. some of them are null.

Let us sketch the procedure for obtaining such relations:

- the expressions of the strain energy in two frames $\mathcal{R}$ and $\mathcal{R}'$ related by a covering operation are

$$V=\frac{1}{2}\{\varepsilon\}^\top[C]\{\varepsilon\},\quad V'=\frac{1}{2}\{\varepsilon'\}^\top[C]\{\varepsilon'\};\tag{59}$$

- the strain tensor $\{\varepsilon'\}$ can be written in the frame $\mathcal{R}$:

$$\{\varepsilon'\}=[R]\{\varepsilon\},\tag{60}$$

  with $[R]$ the orthogonal matrix corresponding to the covering operation, i.e. to the symmetry of the material;
- injecting eq. (60) into V', eq. (59), and putting $V=V'$, gives the equation

$$\{\varepsilon\}^\top[C]\{\varepsilon\}=([R]\{\varepsilon\})^\top[C][R]\{\varepsilon\}\quad\forall\{\varepsilon\};\tag{61}$$

- this unique scalar equation gives all the relations that must be true for the components of $[C]$ because it is true $\forall\{\varepsilon\}$.

This procedure is not the only one; in fact, in place of working with the strain energy, one could directly state that $[C]$, or $[S]$, does not change when passing from $\mathcal{R}$ to $\mathcal{R}'$.

This approach is practically equivalent to the previous one, but it gives a matrix equation. For instance, for $[C]$ we have:

$$\begin{aligned}&\{\sigma\}=[C]\{\varepsilon\},\ \textit{and}\ \{\sigma'\}=[C]\{\varepsilon'\}\ \rightarrow\\ &[R]^\top\{\sigma'\}=[C][R]^\top\{\varepsilon'\}\ \rightarrow\\ &\{\sigma'\}=[R][C][R]^\top\{\varepsilon'\}\ \Rightarrow\ [C]=[R][C][R]^\top.\end{aligned} \tag{62}$$

Now, the question is:

which orthogonal matrix $[R]$ represents a given covering operation?

Before, we need introduce the algebra of rotations for 4^{th}-rank tensors.

Algebra of tensor rotations

Let us consider two orthonormal bases $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\mathcal{B}' = \{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ and let us suppose that these two bases are related by the orthogonal tensor **U**.

This last is not necessarily a proper rotation, because reflections in a plane are possible too.

We define **U** as the tensor such that (since now, we will indicate by a prime ' a component in $\mathcal{B}'$ or a vector or tensor whose components are given in $\mathcal{B}'$)

$$\mathbf{e}_i = \mathbf{U}\mathbf{e}'_i \quad \Rightarrow \quad \mathbf{e}'_i = \mathbf{U}^\top \mathbf{e}_i; \tag{63}$$

with this definition, it is easy to show that

$$\mathbf{U} = \left[\begin{array}{c} \mathbf{e}'_1 \\ \hline \mathbf{e}'_2 \\ \hline \mathbf{e}'_3 \end{array}\right], \quad \mathbf{U}^\top = \left[\begin{array}{c|c|c} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \end{array}\right], \tag{64}$$

i.e. the matrix $[U]$ representing $\mathbf{U}$ in the basis $\mathcal{B}'$ has for rows the components of the vectors of $\mathcal{B}'$ expressed in the basis $\mathcal{B}$.

Algebraically, the components of $\mathbf{U}$ are the director cosines of the angles between two corresponding axes in $\mathcal{B}$ and $\mathcal{B}'$.

Using the above equations, the components in $\mathcal{B}'$ of a tensor of any rank r can be expressed as a linear combination of its components in $\mathcal{B}$, the coefficients of the combination being products of r components of $\mathbf{U}$:

considering that

$$\mathbf{e}_i = \mathbf{U}\mathbf{e}'_i = U_{pq}(\mathbf{e}'_p \otimes \mathbf{e}'_q)\mathbf{e}'_i = U_{pq}\delta_{qi}\mathbf{e}'_p = U_{pi}\mathbf{e}'_p, \tag{65}$$

then:

- for a vector $(r = 1)$ it is

$$\mathbf{w} = w_i\mathbf{e}_i = w_i U_{ki}\mathbf{e}'_k \quad \Rightarrow \quad w'_k = U_{ki} w_i; \tag{66}$$

- for a second-rank tensor $(r = 2)$ it is

$$\begin{aligned} \mathbf{L} &= L_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = L_{ij} U_{mi}\mathbf{e}'_m \otimes U_{nj}\mathbf{e}'_n = \\ & U_{mi} U_{nj} L_{ij}\mathbf{e}'_m \otimes \mathbf{e}'_n \quad \Rightarrow \quad L'_{mn} = U_{mi} U_{nj} L_{ij}; \end{aligned} \tag{67}$$

- for a 4^{th}-rank tensor ($r = 4$) it is[3]

$$\begin{aligned}\mathbb{E} =& E_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l = \\ & E_{ijkl} U_{mi}\mathbf{e}'_m \otimes U_{nj}\mathbf{e}'_n \otimes U_{pk}\mathbf{e}'_p \otimes U_{ql}\mathbf{e}'_q = \\ & U_{mi} U_{nj} U_{pk} U_{ql} E_{ijkl}\mathbf{e}'_m \otimes \mathbf{e}'_n \otimes \mathbf{e}'_p \otimes \mathbf{e}'_q \quad \Rightarrow \end{aligned} \tag{68}$$

$$E'_{mnpq} = U_{mi} U_{nj} U_{pk} U_{ql} E_{ijkl}.$$

---

[3] $\forall$ **A**, **B** and **L** $\in$Lin, $\mathbf{A} \otimes \mathbf{B}$ is the 4^{th}-rank tensor defined by $(\mathbf{A} \otimes \mathbf{B})\mathbf{L} := (\mathbf{B} \cdot \mathbf{L})\mathbf{A}$. Applying this rule to the dyads of a basis, we get : $(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l)(\mathbf{e}_p \otimes \mathbf{e}_q) = (\mathbf{e}_k \otimes \mathbf{e}_l) \cdot (\mathbf{e}_p \otimes \mathbf{e}_q)(\mathbf{e}_i \otimes \mathbf{e}_j) = \delta_{kp}\delta_{lq}(\mathbf{e}_i \otimes \mathbf{e}_j)$.

An advanced tensor representation for rotations

Introducing new operators it is possible to give a more concise, advanced representation for tensor rotations.

Given **A**, **B** $\in$ Lin[4], the conjugation product $\mathbf{A} \boxtimes \mathbf{B}$ is the fourth-rank tensor defined by the operation

$$(\mathbf{A} \boxtimes \mathbf{B})\mathbf{C} := \mathbf{A}\mathbf{C}\mathbf{B}^\top \quad \forall \mathbf{C} \in \mathrm{Lin}. \tag{69}$$

It is worth to remark that eq. (69) implies that for the vectors of a basis it is

$$(\mathbf{e}_i \otimes \mathbf{e}_j) \boxtimes (\mathbf{e}_k \otimes \mathbf{e}_l) = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_j \otimes \mathbf{e}_l, \tag{70}$$

which implies that

$$(\mathbf{A} \boxtimes \mathbf{B})_{ijkl} = A_{ik} B_{jl}. \tag{71}$$

---

[4] We will indicate by $\mathcal{V}$ the translations vector space, associated with the ordinary Euclidean space $\mathcal{E}$, by Lin the manifold of second-order tensors on $\mathcal{V}$ and by $\mathbb{L}$in that of fourth-rank tensors on Lin.

Once defined the transpose $\mathbb{A}^\top$ of a fourth-rank tensor $\mathbb{A}$ as the unique tensor such that

$$\mathbf{L} \cdot (\mathbb{A}\mathbf{M}) = \mathbf{M} \cdot (\mathbb{A}^\top \mathbf{L}) \quad \forall \mathbf{L}, \mathbf{M} \in \mathrm{Lin}, \tag{72}$$

it is immediate to show that

$$\begin{aligned} (\mathbf{A} \otimes \mathbf{B})^\top &= \mathbf{B} \otimes \mathbf{A}, \\ (\mathbf{A} \boxtimes \mathbf{B})^\top &= \mathbf{A}^\top \boxtimes \mathbf{B}^\top. \end{aligned} \tag{73}$$

Like for tensors in Lin, also a tensor $\mathbb{A} \in \mathbb{L}\mathrm{in}$ is said to be symmetric $\iff \mathbb{A} = \mathbb{A}^\top$. It is simple to check that

$$\mathbb{A} = \mathbb{A}^\top \quad \Rightarrow \quad A^\top_{ijkl} = A_{klij}, \tag{74}$$

i.e., the major symmetries of the elastic tensors $\mathbb{E}$ and $\mathbb{Z}$ actually coincide with the definition of symmetric tensor in $\mathbb{L}\mathrm{in}$.

For an orthogonal second-rank tensor $\mathbf{U}$, we define its orthogonal conjugator $\mathbb{U}$ as

$$\mathbb{U} := \mathbf{U} \boxtimes \mathbf{U}; \tag{75}$$

it is not difficult to show that just as $\mathbf{U}$ preserves scalar products of elements in $\mathcal{V}$, its associated orthogonal conjugator $\mathbb{U}$ preserves scalar products in Lin:

$$\mathbb{U}\mathbf{A} \cdot \mathbb{U}\mathbf{B} = \mathbf{A} \cdot \mathbf{B} \quad \forall \mathbf{A}, \mathbf{B} \in \mathrm{Lin}. \tag{76}$$

In other words, $\mathbb{U}$ is an orthogonal tensor in $\mathbb{L}$in. Introducing the identity of $\mathbb{L}$in,

$$\mathbb{I} := \mathbf{I} \boxtimes \mathbf{I} \quad \Rightarrow \quad \mathbb{I} = I_{ijkl}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l) = \delta_{ik}\delta_{jl}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l), \tag{77}$$

it is easy to recognize that also for rotations in $\mathbb{L}$in

$$\mathbb{U}\mathbb{U}^\top = \mathbb{U}^\top\mathbb{U} = \mathbb{I}. \tag{78}$$

Be $\mathbf{n} \in \mathcal{V}$ a unit vector and let us suppose that $\mathbf{n}$ is orthogonal to a symmetry plane. Then

$$\mathbf{U} := \mathbf{I} - 2\mathbf{n} \otimes \mathbf{n}, \quad \mathbf{U} = \mathbf{U}^\top, \quad \mathbb{U} = \mathbf{U} \boxtimes \mathbf{U} = \mathbb{U}^\top, \tag{79}$$

is the orthogonal tensor describing the symmetry in the plane whose normal is $\mathbf{n}$. In fact,

$$\begin{aligned} &\mathbf{U}\mathbf{n} = -\mathbf{n} \\ &\mathbf{U}\mathbf{m} = \mathbf{m} \quad \forall \mathbf{m} \in \mathcal{V} : \ \mathbf{m} \cdot \mathbf{n} = 0, \ |\mathbf{m}| = 1. \end{aligned} \tag{80}$$

It is now possible to give a compact form to results (66), (67) and (68):

$$\begin{aligned} &\mathbf{w}' = \mathbf{U}\mathbf{w}, \\ &\mathbf{L}' = \mathbf{U}\mathbf{L}\mathbf{U}^\top = (\mathbf{U} \boxtimes \mathbf{U})\mathbf{L} = \mathbb{U}\mathbf{L}, \\ &\mathbb{E}' = (\mathbf{U} \boxtimes \mathbf{U})\mathbb{E}(\mathbf{U} \boxtimes \mathbf{U})^\top = \mathbb{U}\mathbb{E}\mathbb{U}^\top. \end{aligned} \tag{81}$$

The Kelvin representation of a rotation tensor

Using eq. (55) and the result of eq. (71), we obtain the matrix $[R]$ that corresponds, in the Kelvin's notation, to tensor $\mathbb{U}$:

$$[R] = \begin{bmatrix} U_{11}^2 & U_{12}^2 & U_{13}^2 & \sqrt{2}U_{12}U_{13} & \sqrt{2}U_{13}U_{11} & \sqrt{2}U_{11}U_{12} \\ U_{21}^2 & U_{22}^2 & U_{23}^2 & \sqrt{2}U_{22}U_{23} & \sqrt{2}U_{23}U_{21} & \sqrt{2}U_{21}U_{22} \\ U_{31}^2 & U_{32}^2 & U_{33}^2 & \sqrt{2}U_{32}U_{33} & \sqrt{2}U_{33}U_{31} & \sqrt{2}U_{31}U_{32} \\ \sqrt{2}U_{21}U_{31} & \sqrt{2}U_{22}U_{32} & \sqrt{2}U_{23}U_{33} & U_{23}U_{32} + U_{22}U_{33} & U_{33}U_{21} + U_{31}U_{23} & U_{31}U_{22} + U_{32}U_{21} \\ \sqrt{2}U_{31}U_{11} & \sqrt{2}U_{32}U_{12} & \sqrt{2}U_{33}U_{13} & U_{32}U_{13} + U_{33}U_{12} & U_{31}U_{13} + U_{33}U_{11} & U_{31}U_{12} + U_{32}U_{11} \\ \sqrt{2}U_{11}U_{21} & \sqrt{2}U_{12}U_{22} & \sqrt{2}U_{13}U_{23} & U_{12}U_{23} + U_{13}U_{22} & U_{11}U_{23} + U_{13}U_{21} & U_{11}U_{22} + U_{12}U_{21} \end{bmatrix} \tag{82}$$

The above matrix $[R]$ allows for the change of basis of any second-rank tensor in the Kelvin's notation.

In particular for $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$:

$$\{\sigma\}' = [R]\{\sigma\}, \quad \{\varepsilon\}' = [R]\{\varepsilon\}. \tag{83}$$

It can be checked that, when **U** is an orthogonal tensor, then

$$[R][R]^\top = [R]^\top[R] = [I], \tag{84}$$

i.e., $[R]$ is an orthogonal matrix in $\mathbb{R}^6$; this is not the case with the Voigt's notation.

Hence, $[R]$ represents, in the given basis, an orthogonal tensor of Lin over a manifold of $\mathbb{R}^6$.

It is impossible to put the result of eq. (68) in matrix form, because also in the Kelvin's notation it depends upon four suffixes.

Nevertheless, it is of course possible to express all the components of such an operator; unfortunately, in the most general case these components have a so complicate and long expression that it is practically impossible to write down all of them, so they are omitted here.

A tensorial characterization of elastic symmetries

The previous results let characterize in an elegant tensorial form the existence of elastic symmetries in a solid.

Let us suppose that a material has a given elastic symmetry and that the two bases $\mathcal{B}$ and $\mathcal{B}'$ correspond to equivalent directions with respect to the symmetry of concern.

Physically, this means that it is not possible to detect the change from $\mathcal{B}$ to $\mathcal{B}'$ by experiments measuring stresses, because the behavior is exactly the same in the two cases: $\mathbb{E}=\mathbb{E}'$. Then, applying eq. $(81)_2$ to $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$,

$$\boldsymbol{\sigma}'=\mathbb{U}\boldsymbol{\sigma}, \quad \boldsymbol{\varepsilon}'=\mathbb{U}\boldsymbol{\varepsilon}, \tag{85}$$

and the Hooke's law, eq. (40), we get, because $\mathbb{E}=\mathbb{E}'$,

$$\boldsymbol{\sigma}' = \mathbb{E}\boldsymbol{\varepsilon}' \;\rightarrow\; \mathbb{U}\boldsymbol{\sigma} = \mathbb{E}\mathbb{U}\boldsymbol{\varepsilon} \;\rightarrow \mathbb{U}\mathbb{E}\boldsymbol{\varepsilon} = \mathbb{E}\mathbb{U}\boldsymbol{\varepsilon} \;\Rightarrow\; \mathbb{U}\mathbb{E} = \mathbb{E}\mathbb{U}. \quad (86)$$

Hence, an orthogonal transformation **U** is in the elastic symmetry group of the material if and only if $\mathbb{E}$ and $\mathbb{U}$ commute, $\mathbb{U}$ being the orthogonal conjugator of **U**.

The result of eq. (86) constitutes also a way for determining the number and type of independent elastic moduli, i.e. the distinct components of $\mathbb{E}$.

This is the way sketched, with reference to matrix $[C]$, in eq. (62); however, the energetic approach is preferred here.