International Doctorate in Civil and Environmental Engineering

Anisotropic Structures - Theory and Design

Strutture anisotrope: teoria e progetto

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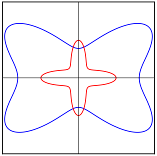
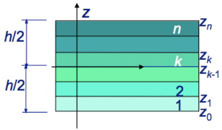


Lesson 6 - May 29, 2019 - DICEA - Universitá di Firenze

Topics of the sixth lesson

- A short introduction to laminated anisotropic structures - Part 2
- Some rules for the general design of laminates

Recall of some basic facts about laminates



$$\left\{\begin{array}{c}\mathbf{N}\\ \hline \mathbf{M}\end{array}\right\} = \left[\begin{array}{c|c} h\mathbb{A} & \frac{h^2}{2}\mathbb{B}\\ \hline \frac{h^2}{2}\mathbb{B} & \frac{h^3}{12}\mathbb{D}\end{array}\right] \left\{\begin{array}{c}\boldsymbol{\varepsilon}^0\\ \hline \boldsymbol{\kappa}\end{array}\right\}, \quad \mathbb{C} = \mathbb{A} - \mathbb{D}. \tag{1}$$

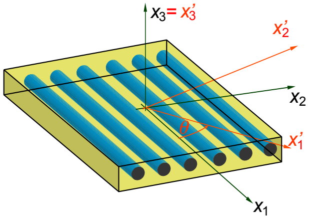
For identical plies,

$$\begin{aligned} \mathbb{A} &= \sum_{k=1}^{n} a_k \mathbb{Q}(\delta_k), \quad \mathbb{B} = \sum_{k=1}^{n} b_k \mathbb{Q}(\delta_k),\\ \mathbb{C} &= \sum_{k=1}^{n} c_k \mathbb{Q}(\delta_k), \quad \mathbb{D} = \sum_{k=1}^{n} d_k \mathbb{Q}(\delta_k), \end{aligned} \tag{2}$$

where

$$\begin{aligned} a_k &= \frac{1}{n}, \quad b_k = \frac{1}{n^2}(2k - n - 1), \quad c_k = a_k - d_k,\\ d_k &= \frac{1}{n^3}\left[12k(k - n - 1) + 4 + 3n(n+2)\right], \end{aligned} \tag{3}$$

With the polar formalism, we get



$$\mathbb{A} \quad \rightarrow \quad \begin{cases} T_0^A = \dfrac{1}{h} \displaystyle\sum_{k=1}^{n} T_{0\,k}(z_k - z_{k-1}) \\ T_1^A = \dfrac{1}{h} \displaystyle\sum_{k=1}^{n} T_{1\,k}(z_k - z_{k-1}) \\ R_0^A e^{4i\Phi_0^A} = \dfrac{1}{h} \displaystyle\sum_{k=1}^{n} R_{0\,k} e^{4i(\Phi_{0\,k}+\delta_k)}(z_k - z_{k-1}) \\ R_1^A e^{2i\Phi_1^A} = \dfrac{1}{h} \displaystyle\sum_{k=1}^{n} R_{1\,k} e^{2i(\Phi_{1\,k}+\delta_k)}(z_k - z_{k-1}) \end{cases} \tag{4}$$

$$
\mathbb{B} \quad \rightarrow \quad \begin{cases} T_0^B = \dfrac{1}{h^2} \displaystyle\sum_{k=1}^{n} T_{0\,k}(z_k^2 - z_{k-1}^2) \\ T_1^B = \dfrac{1}{h^2} \displaystyle\sum_{k=1}^{n} T_{1\,k}(z_k^2 - z_{k-1}^2) \\ R_0^B e^{4i\Phi_0^B} = \dfrac{1}{h^2} \displaystyle\sum_{k=1}^{n} R_{0\,k} e^{4i(\Phi_{0\,k}+\delta_k)}(z_k^2 - z_{k-1}^2) \\ R_1^B e^{2i\Phi_1^B} = \dfrac{1}{h^2} \displaystyle\sum_{k=1}^{n} R_{1\,k} e^{2i(\Phi_{1\,k}+\delta_k)}(z_k^2 - z_{k-1}^2) \end{cases} \tag{5}
$$

$$
\mathbb{D} \quad \to \quad \begin{cases} T_0^D = \dfrac{4}{h^3} \displaystyle\sum_{k=1}^{n} T_{0\,k}(z_k^3 - z_{k-1}^3) \\ T_1^D = \dfrac{4}{h^3} \displaystyle\sum_{k=1}^{n} T_{1\,k}(z_k^3 - z_{k-1}^3) \\ R_0^D e^{4i\Phi_0^D} = \dfrac{4}{h^3} \displaystyle\sum_{k=1}^{n} R_{0\,k} e^{4i(\Phi_{0\,k}+\delta_k)}(z_k^3 - z_{k-1}^3) \\ R_1^D e^{2i\Phi_1^D} = \dfrac{4}{h^3} \displaystyle\sum_{k=1}^{n} R_{1\,k} e^{2i(\Phi_{1\,k}+\delta_k)}(z_k^3 - z_{k-1}^3) \end{cases} \tag{6}
$$

Some remarks:

- the isotropic and anisotropic parts of all the tensors remain separated in the homogenization of the polar parameters, for all the tensors
- it is immediately apparent that special orthotropies are preserved:

$$
\begin{aligned}
R_{0\,k} = 0 \ \forall k \ &\Rightarrow \ R_0^A = R_0^B = R_0^C = R_0^D = 0 \\
R_{1\,k} = 0 \ \forall k \ &\Rightarrow \ R_1^A = R_1^B = R_1^C = R_1^D = 0
\end{aligned}
\tag{7}
$$

More results are obtained for laminates of identical plies ...

The polar method for the case of identical plies

$$\mathbb{A} \to \begin{cases} T_0^A = T_0 \\ T_1^A = T_1 \\ R_0^A e^{4i\Phi_0^A} = R_0 e^{4i\Phi_0}(\xi_1 + i\xi_3) \\ R_1^A e^{2i\Phi_1^A} = R_1 e^{2i\Phi_1}(\xi_2 + i\xi_4) \end{cases} \tag{8}$$

$$\mathbb{B} \to \begin{cases} T_0^B = 0 \\ T_1^B = 0 \\ R_0^B e^{4i\Phi_0^B} = R_0 e^{4i\Phi_0}(\xi_5 + i\xi_7) \\ R_1^B e^{2i\Phi_1^B} = R_1 e^{2i\Phi_1}(\xi_6 + i\xi_8) \end{cases} \tag{9}$$

$$\mathbb{D} \to \begin{cases} T_0^D = T_0 \\ T_1^D = T_1 \\ R_0^D e^{4i\Phi_0^D} = R_0 e^{4i\Phi_0}(\xi_9 + i\xi_{11}) \\ R_1^D e^{2i\Phi_1^D} = R_1 e^{2i\Phi_1}(\xi_{10} + i\xi_{12}) \end{cases} \tag{10}$$

The polar method for the case of identical plies

$$\mathbb{A} \rightarrow \begin{cases} T_0^A = T_0 \\ T_1^A = T_1 \\ R_0^A e^{4i\Phi_0^A} = R_0 e^{4i\Phi_0}(\xi_1 + i\xi_3) \\ R_1^A e^{2i\Phi_1^A} = R_1 e^{2i\Phi_1}(\xi_2 + i\xi_4) \end{cases} \quad (8)$$

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Lamination parameters (Tsai & Pagano, 1968)

$$\begin{cases} \xi_1 + i\xi_3 = \sum_{j=1}^{n} a_j e^{4i\delta_j} \\ \xi_2 + i\xi_4 = \sum_{j=1}^{n} a_j e^{2i\delta_j} \end{cases} \quad (11)$$

$$\begin{cases} \xi_5 + i\xi_7 = \sum_{j=1}^{n} b_j \; e^{4i\delta_j} \\ \xi_6 + i\xi_8 = \sum_{j=1}^{n} b_j \; e^{2i\delta_j} \end{cases} \quad (12)$$

$$\begin{cases} \xi_9 + i\xi_{11} = \sum_{j=1}^{n} d_j \; e^{4i\delta_j} \\ \xi_{10} + i\xi_{12} = \sum_{j=1}^{n} d_j \; e^{2i\delta_j} \end{cases} \quad (13)$$

We then can see that for laminates of identical plies, all what has been said in the general case is still valid and in addition:

- the isotropic part of $\mathbb{A}$ and $\mathbb{D}$ are identical to that of the basic layer
- the isotropic part of $\mathbb{B}$ vanishes: $\mathbb{B}$ is merely anisotropic, with a null mean
- hence, only the anisotropic part of the laminate can be tailored, while the choice of the basic layer automatically fixes the anisotropic part
- this eliminates from the design problem the isotropic part, hence 6 design variables are eliminated

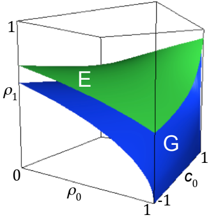
- another decomposition is obtained, that between material and geometric part
- the first one is fixed once the basic layer chosen, and it is represented by the polar parameters of the basic layer
- the second one is that to be designed, it accounts for the geometry of the laminate, i.e. orientation angles and stacking sequence, and is represented by the lamination parameters
- to be remarked that this separation is possible only in the case of identical plies: the lamination parameters cannot be defined for hybrid laminates, but the polar parameters yes, they are universally valid, so they represent a more general concept and tool than that of lamination parameters
- a question arises: can the lamination parameters, or, in the end, the mechanical parameters of the laminate, take any value?

Geometrical bounds

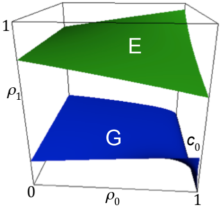
The geometrical bounds (PV, J of Elas, 2013) replace, for laminates, the less strict thermodynamical bounds on elastic constants : laminates belong to a subset of the general elastic materials

$$\rho=\frac{R_0^L}{R_1^L},\quad \tau_0=\frac{T_0^L}{R_0^L},\quad \tau_1=\frac{T_1^L}{R_1^L},\quad \rho_0=\frac{R_0}{R_0^L},\quad \rho_1=\frac{R_1}{R_1^L},\quad c_0=\cos 4\Phi_0.$$

$$\begin{cases} 0\leq\rho_0 \\ 0\leq\rho_1 \\ \rho_0\leq 1 \\ 2\rho_1^2\leq\dfrac{1-\rho_0^2}{1-(-1)^{K^L}\rho_0\, c_0} \end{cases}$$



Carbon-epoxy T-300/5208



Braided carbon-epoxy BR45-a

These bounds concern exclusively the polar parameters of $\mathbb{A}$ or $\mathbb{D}$ separately

Nevertheless, these cannot be completely independent, because obtained as functions of the same quantities: the polar parameters of the basic layer and the orientation angles, besides the stacking sequence

Hence, it should be preferable to establish the admissible set of all the laminate's polar parameters

This is still to be done and probably impossible to be obtained: previous studies on this subject, but done directly on the lamination parameters, so bounded only to the case of laminates of identical plies, never succeeded in solving this still open problem

This point constitutes one of the most serious mathematical open problems for a correct formulation of optimization problems of laminates including at the same time extension and bending properties

For the while there are only two possibilities

- to consider problems concerning only $\mathbb{A}$ or $\mathbb{D}$; this is what is commonly done, and usually only $\mathbb{A}$ is designed
- to bound the research of the solution to the set of quasi-homogenous laminates; this approach is mathematically rigorous, because the admissible set of design variables is strictly the same for $\mathbb{A}$ and $\mathbb{D}$.

Sensitivity to orientation errors

- Errors always inexorably affect each quantity in practical realization. It is hence interesting to ponder what are the effects of such errors.
- In the case of laminates, the most interesting error is that affecting ply orientations. About properties, different choices can be done.
- A simple problem: which is the effect of an orientation error of a single layer on the coupling of a laminate designed to be uncoupled? (PV, J Elas, 2002)
- The problem is stated introducing a suitable measure of the coupling and then analyzing the effects on it of an error on a single layer. The measure is the degree of coupling β defined as

$$\beta = \frac{B}{B_{max}}, \tag{14}$$

where B is a suitable norm of $\mathbb{B}$ and B_{max} is highest possible value. Of course, $\beta \in [0, 1]$ and $\beta = 0$ corresponds to uncoupling, while $\beta = 1$ to the highest possible coupling for the laminate. We take for B

$$B = \sqrt{T_0^{B\,2} + 2T_1^{B\,2} + R_0^{B\,2} + 4R_1^{B\,2}}, \tag{15}$$

For a laminate composed by identical layers

$$B = \sqrt{{R_0^B}^2 + 4{R_1^B}^2}. \tag{16}$$

With some standard passages we get

$$B = \frac{h^2}{2}\sqrt{\begin{array}{l}\left({R_0}^2 + 4{R_1}^2\right)\displaystyle\sum_{j=-p}^{p} b_j^2 + \\ 2\displaystyle\sum_{l=-p}^{p}\sum_{m=l+1}^{p} b_l b_m \left[{R_0}^2\cos 4(\delta_l - \delta_m) + 4{R_1}^2\cos 2(\delta_l - \delta_m)\right]\end{array}} \tag{17}$$

To have B_{max} the term

$$2\sum_{l=-p}^{p}\sum_{m=l+1}^{p} b_l b_m \left[{R_0}^2\cos 4(\delta_l - \delta_m) + 4{R_1}^2\cos 2(\delta_l - \delta_m)\right] \tag{18}$$

must be maximized (the rest depends on the basic layer).

This expression can be rewritten as

$$R_1{}^2 \sum_{l=-p}^{p} \sum_{m=l+1}^{p} b_l b_m \mu(\rho, \delta_{lm}), \quad \delta_{lm} = \delta_l - \delta_m, \tag{19}$$

where

$$\mu(\rho, \delta_{lm}) = \rho^2 \cos 4\delta_{lm} + 4 \cos 2\delta_{lm}. \tag{20}$$

Because coefficients b_j are odd with respect to the mid-plane, $b_l b_m > 0 \iff$ the plies l and m are on the same half of the stack with respect to the mid-plane.

Hence in this case, to maximize B, the function $\mu(\rho, \delta_{lm})$ must take the maximum value for each possible couple of layers, while in the other case of layers on the opposite sides of the mid-plane, it must take the minimum.

Analyzing function $\mu(\rho, \delta_{lm})$, see the figure, one can see that:

- if $\rho \leq 1$, $\mu(\rho, \delta_{lm})$ is maximum for $\delta_{lm} = 0$ and minimum for $\delta_{lm} = \pi/2$;
- if $\rho > 1$, $\mu(\rho, \delta_{lm})$ has a global maximum in $\delta_{lm} = 0$, a local maximum in $\delta_{lm} = \pi/2$ and a minimum for $\delta_{lm} = \delta^*$, where

$$\delta^* = \frac{1}{2} \arccos\left(-\frac{1}{\rho^2}\right). \tag{21}$$

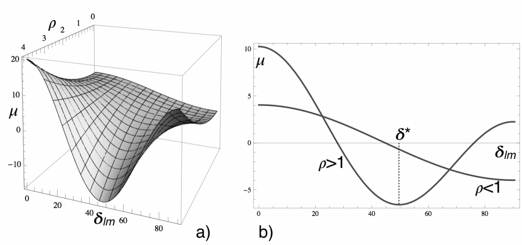


Figure: Function $\mu(\rho, \delta_{lm})$; a) 3D view, b) 2D view for two values of ρ.

$\rightarrow B = B_{max}$ when the mid-plane divides the stacks into two parts, in each one of them the orientation is unique for all the plies and the two orientations differ of the angle $\pi/2$ if $\rho \leq 1$, of δ^* if $\rho > 1$.

The value of B_{max} can then be easily calculated:

$$B_{max} = \begin{cases} 2h^2 R_1 \displaystyle\sum_{j=1}^{p} b_j = \frac{1}{2} h^2 R_1 (n^2 - n \mod 2) \quad \text{if} \quad \rho \leq 1, \\ 2h^2 R_1 \displaystyle\frac{1+\rho^2}{2\rho} \sum_{j=1}^{p} b_j = \frac{1}{2} h^2 R_1 \frac{1+\rho^2}{2\rho} (n^2 - n \mod 2) \quad \text{if} \quad \rho > 1. \end{cases} \tag{22}$$

A similar procedure allows for calculating B_ε, i.e. the norm of $\mathbb{B}$ when only one the orientation of the m-th layer is affected by an orientation error, say ε_m:

$$B_\varepsilon = \frac{1}{\sqrt{2}} h^2 R_1 |b_m| \sqrt{\rho^2 (1 - \cos 4\varepsilon_m) + 4(1 - \cos 2\varepsilon_m)}. \tag{23}$$

Because the coefficients b_j take the highest absolute value for the two external layers, i.e. for $m = p$, we get that the highest value of β is obtained for an error of orientation affecting one of the two outer layers. It is then easy to calculate β:

$$\beta = \begin{cases} \dfrac{\lambda}{\sqrt{2}}\sqrt{\rho^2(1-\cos 4\varepsilon_p) + 4(1-\cos 2\varepsilon_p)} & \text{if} \quad \rho \leq 1, \\ \dfrac{\lambda}{\sqrt{2}}\dfrac{2\rho}{1+\rho^2}\sqrt{\rho^2(1-\cos 4\varepsilon_p) + 4(1-\cos 2\varepsilon_p)} & \text{if} \quad \rho > 1, \end{cases} \tag{24}$$

with

$$\lambda = \frac{1}{2}\frac{b_p}{\sum_{j=1}^{p} b_j} = 2\frac{n-1}{n^2 - \mathrm{mod}2}. \tag{25}$$

The last two equations show that the sensitivity β to a single defect decreases with n and that the orientation angles do not matter. It is then easy to show that the angle that maximizes β is $\pi/2$ if $\rho \leq 1$ and δ^* if $\rho > 1$; in both the cases,

$$\beta_{max} = 2\lambda. \tag{26}$$

In the figure, the function β/λ. Materials with $\rho \to \infty \;\Rightarrow\; R_1 = 0$, i.e. square symmetric layers, are the most sensitive to errors, while the less sensitive are those with $\rho = 0 \;\Rightarrow\; R_0 = 0$, i.e. layers that are R_0-orthotropic.

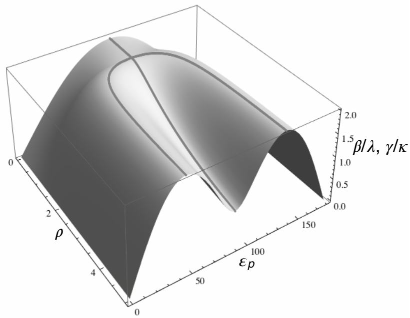


Figure: Function β/λ; the curve is the locus of the stationary values of the function.

Thermal problems

We apply the Hooke-Duhamel constitutive law

$$\boldsymbol{\sigma} = \mathbb{E}(\boldsymbol{\varepsilon} - t\boldsymbol{\alpha}). \tag{27}$$

to laminates; any linear thermal field with a temperature t_{up} and t_{bot}, respectively on the upper and lower surfaces, can be decomposed in the sum of two different fields:

- a constant field of value

$$t = \frac{t_{up} + t_{bot}}{2}; \tag{28}$$

- an antisymmetric linear field characterized by the constant gradient

$$\nabla t = \frac{t_{up} - t_{bot}}{h}, \tag{29}$$

where the temperatures are evaluated with respect to the manufacturing temperature.

Hence, for the $k-$th ply it is

$$\boldsymbol{\sigma}_k = \mathbb{Q}_k(\delta_k)(\boldsymbol{\varepsilon}^0 + x_3\ \boldsymbol{\kappa}) - (t + \nabla t\ x_3)\,\mathbb{Q}(\delta_k)\boldsymbol{\alpha}_k(\delta_k). \qquad (30)$$

Using this equation in **N** and **M**, we get

$$\left\{\frac{\mathbf{N}}{\mathbf{M}}\right\} = \left[\begin{array}{c|c} h\mathbb{A} & \frac{h^2}{2}\mathbb{B} \\ \hline \frac{h^2}{2}\mathbb{B} & \frac{h^3}{12}\mathbb{D} \end{array}\right]\left\{\frac{\boldsymbol{\varepsilon}^0}{\boldsymbol{\kappa}}\right\} - t\left\{\frac{h\mathbf{U}}{\frac{h^2}{2}\mathbf{V}}\right\} - \nabla t\left\{\frac{\frac{h^2}{2}\mathbf{V}}{\frac{h^3}{12}\mathbf{W}}\right\}. \qquad (31)$$

The second-rank symmetric tensors **U**, **V** and **W** are the thermal corresponding of tensors $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{D}$.

- **U**: tensor of in-plane actions per unit of temperature t uniform through the thickness;
- **W**: tensor of bending moments per unit of thermal gradient ∇t;
- **V**: coupling tensor that describes at the same time the in-plane actions produced by $\nabla t = 1$ and the bending moments produced by a field $t = 1$.

$$\mathbf{U} = \frac{1}{h}\sum_{k=1}^{n}\int_{z_{k-1}}^{z_k}\gamma_k(\delta_k)dx_3 = \frac{1}{h}\sum_{k=1}^{n}(z_k - z_{k-1})\gamma_k(\delta_k),$$
$$\mathbf{V} = \frac{2}{h^2}\sum_{k=1}^{n}\int_{z_{k-1}}^{z_k}x_3\gamma_k(\delta_k)dx_3 = \frac{1}{h^2}\sum_{k=1}^{n}(z_k^2 - z_{k-1}^2)\gamma_k(\delta_k), \tag{32}$$
$$\mathbf{W} = \frac{12}{h^3}\sum_{k=1}^{n}\int_{z_{k-1}}^{z_k}x_3^2\gamma_k(\delta_k)dx_3 = \frac{4}{h^3}\sum_{k=1}^{n}(z_k^3 - z_{k-1}^3)\gamma_k(\delta_k).$$

For identical layers

$$\mathbf{U} = \sum_{k=1}^{n} a_k\gamma(\delta_k), \quad \mathbf{V} = \sum_{k=1}^{n} b_k\gamma(\delta_k), \quad \mathbf{W} = \sum_{k=1}^{n} d_k\gamma(\delta_k) \tag{33}$$

with

$$\boldsymbol{\gamma}(\delta_k) = \mathbb{Q}(\delta_k)\boldsymbol{\alpha}(\delta_k), \quad k = 1, ..., n, \tag{34}$$

Inverse law:

$$\left\{\frac{\boldsymbol{\varepsilon}^0}{\boldsymbol{\kappa}}\right\} = \left[\begin{array}{c|c} \frac{1}{h}\mathcal{A} & \frac{2}{h^2}\mathcal{B} \\ \hline \frac{2}{h^2}\mathcal{B}^\top & \frac{12}{h^3}\mathcal{D} \end{array}\right] \left\{\frac{\mathbf{N}}{\mathbf{M}}\right\} + t\left\{\frac{\mathbf{u}}{\mathbf{v}_1}\right\} + \nabla t\left\{\frac{\mathbf{v}_2}{\mathbf{w}}\right\}, \qquad (35)$$

$$\begin{aligned}
\mathbf{u} &= \mathcal{A}\mathbf{U} + \mathcal{B}\mathbf{V} = (\mathbb{A} - 3\mathbb{B}\mathbb{D}^{-1}\mathbb{B})^{-1}(\mathbf{U} - 3\mathbb{B}\mathbb{D}^{-1}\mathbf{V}),\\
\mathbf{v}_1 &= \frac{2}{h}(\mathcal{B}^\top\mathbf{U} + 3\mathcal{D}\mathbf{V}) = \frac{6}{h}(\mathbb{D} - 3\mathbb{B}\mathbb{A}^{-1}\mathbb{B})^{-1}(\mathbf{V} - \mathbb{B}\mathbb{A}^{-1}\mathbf{U}),\\
\mathbf{v}_2 &= \frac{h}{6}(3\mathcal{A}\mathbf{V} + \mathcal{B}\mathbf{W}) = \frac{h}{2}(\mathbb{A} - 3\mathbb{B}\mathbb{D}^{-1}\mathbb{B})^{-1}(\mathbf{V} - \mathbb{B}\mathbb{D}^{-1}\mathbf{W}),\\
\mathbf{w} &= \mathcal{B}^\top\mathbf{V} + \mathcal{D}\mathbf{W} = (\mathbb{D} - 3\mathbb{B}\mathbb{A}^{-1}\mathbb{B})^{-1}(\mathbf{W} - 3\mathbb{B}\mathbb{A}^{-1}\mathbf{V}).
\end{aligned} \qquad (36)$$

Only apparently the inverse law depends upon four tensors, because

$$\mathbf{v}_2 = \frac{h^2}{12}\mathcal{A}\left[\mathcal{D}^{-1}\mathbf{v}_1 + \frac{6}{h}\mathbb{B}(\mathbb{A}^{-1}\mathbf{U} - \mathbb{D}^{-1}\mathbf{W})\right]. \qquad (37)$$

Physical meaning of the above second-rank tensors:

- $\mathbf{u}$ is the tensor of the coefficients of thermal expansion of the laminate for a uniform change of temperature t; its SI units are $^\circ\mathrm{C}^{-1}$;
- $\mathbf{v}_1$ is the tensor of the coefficients of thermal expansion of the laminate for a gradient of temperature ∇t; its SI units are $(m\ ^\circ\mathrm{C})^{-1}$;
- $\mathbf{v}_2$ is the tensor of the coefficients of thermal curvature of the laminate for a uniform change of temperature t; its SI units are $m\ ^\circ\mathrm{C}^{-1}$;
- $\mathbf{w}$ is the tensor of the coefficients of thermal curvature of the laminate for a gradient of temperature ∇t; its SI units are $^\circ\mathrm{C}^{-1}$.

Tensors **U**, **V** and **W** can be expressed using the polar formalism. If T^γ, R^γ and Φ^γ denote the polar components of tensor γ, then

$$\mathbf{U} \rightarrow \begin{cases} T^U = \dfrac{1}{h}\displaystyle\sum_{k=1}^{n} T_k^\gamma (z_k - z_{k-1}), \\ R^U e^{2i\Phi^U} = \dfrac{1}{h}\displaystyle\sum_{k=1}^{n} R_k^\gamma e^{2i(\Phi_k^\gamma + \delta_k)}(z_k - z_{k-1}); \end{cases} \tag{38}$$

$$\mathbf{V} \rightarrow \begin{cases} T^V = \dfrac{1}{h^2}\displaystyle\sum_{k=1}^{n} T_k^\gamma (z_k^2 - z_{k-1}^2), \\ R^V e^{2i\Phi^V} = \dfrac{1}{h^2}\displaystyle\sum_{k=1}^{n} R_k^\gamma e^{2i(\Phi_k^\gamma + \delta_k)}(z_k^2 - z_{k-1}^2); \end{cases} \tag{39}$$

$$\mathbf{W} \rightarrow \begin{cases} T^W = \dfrac{4}{h^3}\displaystyle\sum_{k=1}^{n} T_k^\gamma (z_k^3 - z_{k-1}^3), \\ R^W e^{2i\Phi^W} = \dfrac{4}{h^3}\displaystyle\sum_{k=1}^{n} R_k^\gamma e^{2i(\Phi_k^\gamma + \delta_k)}(z_k^3 - z_{k-1}^3). \end{cases} \tag{40}$$

$\mathbb{B}=\mathbb{O}$, is not sufficient to ensure also thermal-elastic uncoupling. If $\mathbb{B}=\mathbb{O}$ but $\mathbf{V}\neq\mathbf{O}$, we get

$$\mathbf{u}=\mathbb{A}^{-1}\mathbf{U},\quad \mathbf{v}_1=\frac{6}{h}\mathbb{D}^{-1}\mathbf{V},\quad \mathbf{v}_2=\frac{h}{2}\mathbb{A}^{-1}\mathbf{V},\quad \mathbf{w}=\mathbb{D}^{-1}\mathbf{W}. \tag{41}$$

Of course, if also $\mathbf{V}=\mathbf{O}$, then we get immediately that $\mathbf{v}_1=\mathbf{v}_2=\mathbf{O}$, i.e. a laminate is thermally uncoupled $\iff$ $\mathbb{B}=\mathbb{O}$ *and* $\mathbf{V}=\mathbf{O}$.

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For the case of identical layers,

$$\mathbf{U} \rightarrow \begin{cases} T^U = T^\gamma, \\ R^U e^{2i\Phi^U} = R^\gamma e^{2i\Phi^\gamma}(\xi_2 + i\xi_4); \end{cases} \tag{42}$$

$$\mathbf{V} \rightarrow \begin{cases} T^V = 0, \\ R^V e^{2i\Phi^V} = R^\gamma e^{2i\Phi^\gamma}(\xi_6 + i\xi_8); \end{cases} \tag{43}$$

$$\mathbf{W} \rightarrow \begin{cases} T^W = T^\gamma, \\ R^W e^{2i\Phi^W} = R^\gamma e^{2i\Phi^\gamma}(\xi_{10} + i\xi_{12}). \end{cases} \tag{44}$$

Now,

$$\mathbb{B} = \mathbb{O} \ \Rightarrow \ \xi_6 + i\xi_8 = 0 \ \Rightarrow \ \mathbf{V} = \mathbf{O} \ \Rightarrow \ \mathbf{v}_1 = \mathbf{v}_2 = \mathbf{O}. \tag{45}$$

So, for the case of identical plies elastic uncoupling gives also, automatically, thermal uncoupling.

The converse is not true: because $\xi_6 + i\xi_8 = 0 \nRightarrow \xi_5 + i\xi_7 = 0$, $\mathbf{V} = \mathbf{O} \nRightarrow \mathbb{B} = \mathbb{O} \Rightarrow \exists$ laminates such that $\mathbb{B} \neq \mathbb{O}$ and $\mathbf{V} = \mathbf{O} \ \Rightarrow$

$$\begin{aligned}
\mathbf{u} &= \mathcal{A}\mathbf{U} = (\mathbb{A} - 3\mathbb{B}\mathbb{D}^{-1}\mathbb{B})^{-1}\mathbf{U},\\
\mathbf{v}_1 &= \frac{2}{h}\mathcal{B}^\top\mathbf{U} = -\frac{6}{h}(\mathbb{D} - 3\mathbb{B}\mathbb{A}^{-1}\mathbb{B})^{-1}\mathbb{B}\mathbb{A}^{-1}\mathbf{U},\\
\mathbf{v}_2 &= \frac{h}{6}\mathcal{B}\mathbf{W} = -\frac{h}{2}(\mathbb{A} - 3\mathbb{B}\mathbb{D}^{-1}\mathbb{B})^{-1}\mathbb{B}\mathbb{D}^{-1}\mathbf{W},\\
\mathbf{w} &= \mathcal{D}\mathbf{W} = (\mathbb{D} - 3\mathbb{B}\mathbb{A}^{-1}\mathbb{B})^{-1}\mathbf{W},
\end{aligned} \tag{46}$$

i.e. $\mathbf{v}_1$ and $\mathbf{v}_2$ do not vanish necessarily: a change of temperature t makes the plate warp but does not produce bending moments and a gradient ∇t stretches the plate but does not give rise to membrane forces.

More interesting is the use of coupled thermally stable laminates, i.e. of laminates such that (PV, J Elas, 2013)

$$\mathbb{B} \neq \mathbb{O}, \mathbf{v}_1 = \mathbf{O} \text{ and/or } \mathbf{v}_2 = \mathbf{O}. \tag{47}$$

The most important case is that of warp-free laminates: $\mathbf{v}_1 = \mathbf{O}$: a uniform change of temperature, namely the cooling of the laminate from the manufacturing temperature, engenders curvatures:

$$\mathbf{v}_1 = \frac{2}{h}(\mathcal{B}^\top \mathbf{U} + 3\mathcal{D}\mathbf{V}) = \frac{6}{h}(\mathbb{D} - 3\mathbb{B}\mathbb{A}^{-1}\mathbb{B})^{-1}(\mathbf{V} - \mathbb{B}\mathbb{A}^{-1}\mathbf{U}) = \mathbf{O}. \tag{48}$$

It can be shown that different solutions of warp-free laminates can exist; namely,

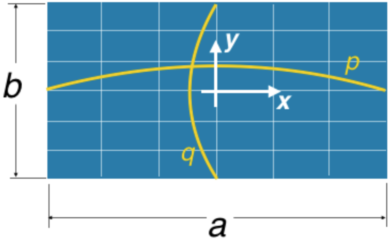
$$R_1 = 0, R_1^A = R_1^B = 0 \tag{49}$$

are two sufficient conditions for warp-free laminates (PV, J Elas, 2013).

Interaction between geometry and anisotropy

Geometry and anisotropy interacts: the concept of anisotropic behavior is an absolute one, but that of anisotropic response no: it depends upon geometry.

We can see that on a simple case, the flexural behavior of a rectangular uncoupled orthotropic laminate:



- Compliance:

$$J = \frac{\gamma_{pq}}{p^4 h^3 (1+\chi^2)^2 \sqrt{R_0^2 + R_1^2}} \frac{1}{\varphi(\xi_9, \xi_{10})};$$

- Buckling load multiplier for the mode pq:

$$\lambda_{pq} = \frac{\pi^2 p^2 h^3}{12 a^2} \frac{(1+\chi^2)^2 \sqrt{R_0^2 + R_1^2}}{N_x + N_y \chi^2} \varphi(\xi_9, \xi_{10});$$

- Frequency of the transversal vibration for the mode pq

$$\omega_{pq}^2 = \frac{\pi^4 p^4 h^3}{12 \mu a^4} (1+\chi^2)^2 \sqrt{R_0^2 + R_1^2} \varphi(\xi_9, \xi_{10}).$$

Wavelengths ratio

$$\chi = \frac{a}{b}\frac{q}{p}$$

Isotropy-to-anisotropy ratio

$$\tau = \frac{T_0 + 2T_1}{\sqrt{R_0^2 + R_1^2}}$$

Anisotropy ratio

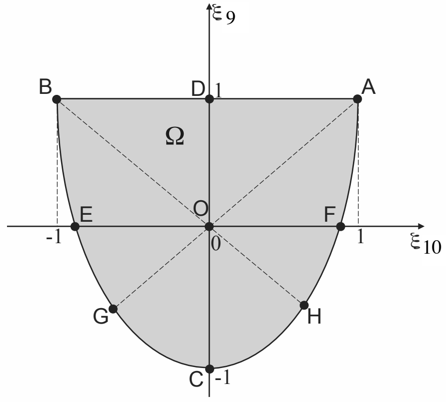
$$\rho = \frac{R_0}{R_1}$$

Lamination parameters

$$\xi_9 = \frac{1}{n^3}\sum_{j=1}^{n} d_j \cos 4\delta_j$$

$$\xi_{10} = \frac{1}{n^3}\sum_{j=1}^{n} d_j \cos 2\delta_j$$

$$d_j = 12j(j-n-1) + 4 + 3n(n+2).$$



$$\varphi(\xi_9,\xi_{10})=\tau+\frac{1}{\sqrt{1+\rho^2}}\left[(-1)^k\rho\xi_9\frac{\chi^4-6\chi^2+1}{(1+\chi^2)^2}+4\xi_{10}\frac{1-\chi^2}{1+\chi^2}\right]$$

$\varphi(\xi_9,\xi_{10})=\tau$ is possible $\rightarrow$ the plate has an isotropic response

Possible solutions:

- $\rho=0,\ \chi=1$; e.g. $p=q$ on square plates of R_0-orthotropic plies
- $\xi_9=0,\ \chi=1$; still $p=q$ and lamination point on line EF: $[\pm(\pi/8)_{n/4},\pm(3\pi/8)_{n/4}]$
- $\rho=\infty,\ \chi=\sqrt{2}\pm1$; plates of square symmetric layers with $\frac{a}{b}=\frac{p}{q}\left(\sqrt{2}\pm1\right)$.
- $\xi_{10}=0,\ \chi=\sqrt{2}\pm1$; still $\frac{a}{b}=\frac{p}{q}\left(\sqrt{2}\pm1\right)$ and lamination point on line CD: $[0_{n/4},(\pi/2)_{n/4},\pm(\pi/4)_{n/4}]$ (*generalized quasi-isotropic laminates*)
- more generally, $\xi_9=\frac{4}{(-1)^K\rho}\frac{\chi^4-1}{\chi^4-6\chi^2+1}\xi_{10}$

Some rules for the general design of laminates

Types of laminates

In industrial applications, some kinds of laminates are used more frequently

They have some special properties normally desired by designers or they automatically offer some advantages

All these rules can be applied only to laminates with identical plies

In the following we will refer to this case

In the design of hybrid laminates, general rules do not exist, apart those seen before and given by the use of the polar formalism

One should consider that the design of an anisotropic laminate normally means:

- the design of the number of plies
- the choice of the material
- the usual design of mechanical facts, like buckling, strength and so on
- the design of the material properties of $\mathbb{A}$ and $\mathbb{D}$
- this design implies that of the elastic symmetries: orthotropy or isotropy or square symmetry etc.
- normally, the uncoupling requirement: $\mathbb{B} = 0$

This shows how much complicate can be the design of laminates and why engineers have since the beginning simplified using only some categories of laminates

Unfortunately, the drawback is that in this way, a solution is almost never an optimal one!

In the following, we will use the polar approach, more effective to determine general properties

Uncoupling: $\mathbb{B}=0$

Uncoupling is the most frequent requirement asked for by designers

Because $T_0^B=T_1^B=0$, a laminate is uncoupled $\iff$

$$R_0^B=R_1^B=0 \tag{50}$$

This happens $\iff$

$$\xi_5=\xi_6=\xi_7=\xi_8=0 \tag{51}$$

A particular choice of the orientations δ_k can hence solve the problem

Nevertheless, there is another possibility, independent from the δ_k

In fact,

$$b_k=\frac{1}{n^2}(2k-n-1)\ \Rightarrow\ \sum_{k=1}^{n} b_k=0. \tag{52}$$

In addition, it can be checked very easily that the b_k are odd with respect to the midplane: if the plies are numbered starting from the midplane, then

$$b_k = \frac{2k}{n^2} \text{ if } n = 2p+1, \;\; b_k = \frac{1}{n^2}(2k - \frac{k}{|k|}), \; b_0 = 0 \text{ if } n = 2p. \quad (53)$$

Hence

$$b_{-k} = -b_k \quad (54)$$

So, a sufficient condition for having $\mathbb{B} = 0$ is to use symmetric stacks:

$$\delta_{-k} = \delta_k \;\; \forall k \quad (55)$$

This is the rule usually adopted by engineers, but contrarily to what is commonly said, this is not a necessary condition for uncoupling: unsymmetric uncoupled laminates really do exist

Quasi-homogeneous laminates

A laminate is quasi-homogeneous $\iff$

$$\mathbb{B} = \mathbb{C} = 0 \tag{56}$$

The polar representation of $\mathbb{C}$ is

$$\mathbb{C} \quad \rightarrow \quad \begin{cases} T_0^C = 0 \\ T_1^C = 0 \\ R_0^C e^{4i\Phi_0^C} = R_0 e^{4i\Phi_0}(\xi_{13} + i\xi_{15}) \\ R_1^C e^{2i\Phi_1^C} = R_1 e^{2i\Phi_1}(\xi_{14} + i\xi_{16}) \end{cases} \tag{57}$$

where

$$\begin{cases} \xi_{13}+i\xi_{15}=\sum_{j=1}^{n} c_j \; e^{4i\delta_j} \\ \xi_{14}+i\xi_{16}=\sum_{j=1}^{n} c_j \; e^{2i\delta_j} \end{cases} \tag{58}$$

Like for $\mathbb{B}$, the isotropic part of $\mathbb{C}$ is null

So, a laminate will be quasi homogeneous $\iff$

$$\xi_5 = \xi_6 = \xi_7 = \xi_8 = \xi_{13} = \xi_{14} = \xi_{15} = \xi_{16} = 0 \tag{59}$$

To remark that by a direct Cartesian approach one should find 12 conditions!

Numbering the plies from the mid-plane gives

$$\begin{aligned} d_k &= \frac{1}{n^3}(12k^2+1) \text{ if } n = 2p+1, \\ d_k &= \frac{1}{n^3}(12k^2-12|k|+4),\ d_0 = 0 \text{ if} n = 2p \rightarrow \end{aligned} \tag{60}$$

$$\begin{aligned} c_k &= \frac{4}{n^3}(p^2+p-3k^2) \text{ if } n = 2p+1, \\ c_k &= \frac{4}{n^3}(p^2-3k^2+3|k|-1),\ c_0 = 0 \text{ if } n = 2p. \end{aligned} \tag{61}$$

It is easily checked that

$$c_{-k} = c_k, \quad \sum_{k=1}^{n} c_k = 0. \tag{62}$$

Quasi-trivial solutions

The conditions (59) can be satisfied by particular sets of the δ_k

Nevertheless, because the sum of the b_k and of the c_k is null, another possibility exist, independent from the δ_k

In fact, it is sufficient to choose the same orientation for a group of layers whose sum of the coefficients b_k and c_k is null

Such a group is called a saturated group; if a sequence is composed by saturated groups, automatically it is quasi-homogeneous, regardless of the value of the orientation of each saturated group

Such solutions are called quasi-trivial, to recall that there is no need to solve equations (59) (PV, GV, IJSS, 2001)

To remark that the search for quasi-trivial solutions needs the satisfaction of 2 integer conditions, instead of the 8 real ones of eq. (59):

$$\sum_{g=1}^{sg}(\sum_{k=1}^{n_k} b_k) = 0, \quad \sum_{g=1}^{sg}(\sum_{k=1}^{n_k} c_k) = 0 \tag{63}$$

where sg is the number of saturated groups and n_k the number of plies in the $g-$th saturated group

Saturated groups can be found by an enumeration algorithm; what is interesting is the high number of quasi-trivial solutions, see the following table

This type of solutions exist also for the uncoupled laminates, which simply must fulfill only eq. $(63)_1$

What is surprising is that the number of un-symmetric solutions is much greater than that of the symmetric ones (the symmetric solution is a special case of the quasi-trivial set)

N. plis	2 g. s.	3 g. s.	4 g. s.	5 g. s.	6 g. s.	Total
7	1 (1)	-	-	-	-	1 (1)
8	1	-	-	-	-	1
9	-	-	-	-	-	-
10	-	-	-	-	-	-
11	3 (2)	-	-	-	-	3 (2)
12	1	-	-	-	-	1
13	2	2	-	-	-	4
14	-	2 (1)	-	-	-	2 (1)
15	2	2	-	-	-	4
16	5	3 (1)	-	-	-	8 (1)
17	15	8	-	-	-	23
18	-	5	-	-	-	5
19	30	22	-	-	-	52
20	30	9	1	-	-	40
21	31	13 (2)	-	-	-	44 (2)
22	17 (2)	98 (1)	13	2	-	130 (3)
23	95 (1)	499	-	-	-	594 (1)
24	140	26	1	-	-	167
25	163	2132	57	-	-	2352
26	54	1059 (2)	354 (3)	26 (2)	2	1495 (7)
27	86 (1)	918	256	21	1	1282 (1)
28	203	4789 (1)	871 (2)	33	6	5902 (3)
29	61	37747	7546	87	-	45441
30	53	5552	512 (3)	29	-	6146 (3)

Figure: Number of quasi-homogenous laminates of the quasi-trivial type; in brackets, the number of symmetric solutions; g.s.: saturated groups

N. plis	2 g. s.	3 g. s.	4 g. s.	5 g. s.	6 g. s.	7 g. s.	8 g. s.	9 g. s.	Total
4	1	-	-	-	-	-	-	-	1
5	-	1	-	-	-	-	-	-	1
6	-	1	-	-	-	-	-	-	1
7	-	1	1	-	-	-	-	-	2
8	1	-	1	-	-	-	-	-	2
9	-	1	2	1	-	-	-	-	4
10	-	4	-	1	-	-	-	-	5
11	-	-	6	4	1	-	-	-	11
12	1	4	9	-	1	-	-	-	15
13	-	-	14	20	6	1	-	-	41
14	-	22	17	17	-	1	-	-	57
15	-	-	5	111	48	9	1	-	174
16	-	29	168	48	29	-	1	-	275
17	-	-	1	458	471	90	12	1	1033
18	-	57	746	686	104	45	-	1	1639

Figure: Number of uncoupled laminates of the quasi-trivial type; g.s.: saturated groups

Just some examples of quasi-homogeneous laminates of the quasi-trivial set

- 8 layers: [0 1 1 0 1 0 0 1]
- 12 layers: [0 1 0 1 1 1 0 0 0 1 0 1]
- 16 layers: [0 01 1 1 1 0 01 1 0 0 0 0 1 1]
- 18 layers: [0 1 2 0 1 2 2 2 1 1 0 0 1 0 0 2 2 1]
  [0 1 2 2 0 1 1 2 0 1 2 0 2 0 1 0 1 2]
  [0 1 2 2 0 1 1 2 0 2 1 0 1 0 2 0 2 1]
  [0 1 2 2 1 0 2 0 1 1 0 2 1 2 0 0 2 1]
  [0 1 1 2 2 2 0 0 2 1 0 0 1 1 1 2 2 0]

All these examples are un-symmetric stacking sequences.

Othotropic laminates

Normally, completely anisotropic laminates are not used by designers: usually orthotropy is a minimal requirement for the elastic symmetries of the laminate

Nevertheless, while it is rather simple to obtain an orthotropic $\mathbb{A}$, it is much more difficult to get orthotropy for $\mathbb{D}$

This is due essentially to the fact that the coefficients d_k change layer by layer, while the a_k are constant, this implying, as already said, that the stacking sequence influences the properties of $\mathbb{D}$, but not of $\mathbb{A}$

For this reason, engineers use some special sequences ensuring the orthotropy of $\mathbb{A}$ but not, generally speaking, of $\mathbb{D}$

This problem is completely forgotten, to such a point that several engineers think that it does not exist at all!

In fact, the almost totality of papers on vibrations and buckling of laminates, where the orthotropy of $\mathbb{D}$ matters, are not correct

In some texts, it is said that if $n > 6$, than the difference with respect to an exact solution, i.e. where $\mathbb{D}$ is really orthotropic, is negligible; accurate studies show that this is false (AV, PV, RA, Mech Adv Mat Struct, 2013)

We consider here, first, the classical rules for obtaining in-plane orthotropy, then some strategies for the bending or the full orthotropy.

Balanced laminates

A laminate is said to be balanced if $\forall$ ply at the orientation δ there is another ply at the orientation $-\delta$

Balanced laminates have hence an even number of plies

The interest in using balanced laminates is that they are orthotropic in extension, regardless of the values of the orientation of the twin layers

In fact, $\mathbb{A}$ is ordinarily orthotropic $\iff$

$$\Phi_0^A - \Phi_1^A = K^A \frac{\pi}{4} \Rightarrow \xi_3 = \xi_4 = 0. \tag{64}$$

The last condition is immediately get operating a rotation of Φ_1^A and remembering how this is done in the polar method

So, finally

$$
\begin{aligned}
\xi_3 = 0 \ &\rightarrow \ \sum_{k=1}^{n} \sin 4\delta_k = 0 \\
\xi_4 = 0 \ &\rightarrow \ \sum_{k=1}^{n} \sin 2\delta_k = 0
\end{aligned}
\tag{65}
$$

Of course, a sufficient condition for fulfilling these two equations is that for each orientation $\delta_k = \alpha$ there is another orientation $\delta_j = -\alpha, \ j \neq k$

This is just a balanced stacking sequence

A particular case of balanced laminates is that of the angle-ply sequences, that have only two possible orientations: α and $-\alpha$

Engineers normally use these sequences in a symmetric stack, so as to get the two main properties: uncoupling and extension orthotropy

Bending orthotropy, however, is not obtained, because of the presence of the coefficients d_k in the expressions of ξ_{11} and ξ_{12}

Because these coefficients are symmetric with respect to the midplane, if the twin layers $\alpha/-\alpha$ are disposed symmetrically with respect to the midplane, to form hence an antisymmetric stack, the laminate will be orthotropic also in bending, but it will be coupled

Because the primary requirement is uncoupling, this kind of strategy is never applied, but in some special cases, see further

Cross-ply laminates

Cross-ply are laminates with only 2 orientations: 0° and 90°

It is immediate to check that in such a case it is

$$\xi_3 = \xi_4 = \xi_{11} = \xi_{12} = 0 \tag{66}$$

The laminate is hence orthotropic at the same time in extension and in bending; once more, normally the sequences are symmetric

Common plywood is a cross-ply laminate

In addition, for an equal number of plies at 0° and 90°,

$$\xi_2 = \sum_{k=1}^{n} a_k \cos 2\delta_k = 0 \;\Rightarrow\; R_1^A = 0 \tag{67}$$

i.e., the laminate is square symmetric in extension (but not in bending, still for the presence of the coefficients d_k)

Quasi-isotropic sequences

Quasi-isotropic sequences are laminates where the only allowed orientations are $0^\circ, \pm 45^\circ, 90^\circ$, with the constraint to have the same number of plies at $+45^\circ$ and -45°

They are called in this way because if the number of the plies in each direction is constant, then $\mathbb{A}$ is isotropic, see further

These laminates are very used in aeronautics, because they have an elastic behavior without weak directions of stiffness, more or less

Also, the presence of fibers in these four directions helps in preventing the propagation of cracks

Of course, being the superposition of a cross-ply and an angle-ply, $\mathbb{A}$ is orthotropic, but not $\mathbb{D}$

Normally, symmetric sequences are used to get uncoupling

Other strategies for orthotropy

We have seen that bending orthotropy cannot be get with the usual strategies

Other strategies can then be used to obtain it or even to get fully orthotropic laminates

We will see in the next lesson that it is possible to formulate a general numerical strategy to obtain laminates with prescribed properties

Here, we bound ourselves to some particular strategies that do not need a numerical solution

A first strategy is to use antisymmetric sequences, that ensure automatically the orthotropy of $\mathbb{A}$ and $\mathbb{D}$, as already said

Then, the other fundamental property, uncoupling, is searched, because no more automatically ensured by the symmetry of the sequence

It can be shown (EV, PV, Comp Struct, 2005) that for an antisymmetric sequence, the condition $\mathbb{B}=0$ can be reduced to the equations ($n=2p$ or $n=2p+1$)

$$\begin{gathered}\left(\sum_{k=2}^{p} b_k \sin 2\delta_k\right)^4 - b_1^2\left(\sum_{k=2}^{p} b_k \sin 2\delta_k\right)^2 + \\ b_1^2\left(\sum_{k=2}^{p} b_k \sin 2\delta_k \cos 2\delta_k\right)^2 = 0, \quad \text{and with} \\ \sin 2\delta_1 = -\frac{1}{b_1}\sum k = 2^p b_k \sin 2\delta_k.\end{gathered} \tag{68}$$

The above equation is highly non linear and it can be solved only numerically

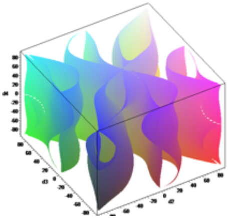
Just two examples, whose solution loci are depicted in the figures

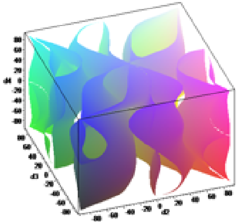
8 plies

$$(5\sin 2\delta_2 + 2\sin 2\delta_3 + \sin 2\delta_4)^4 - 49(5\sin 2\delta_2 + 2\sin 2\delta_3 + \sin 2\delta_4)^2 + 49(5\sin 2\delta_2 \cos 2\delta_2 + 2\sin 2\delta_3 \cos 2\delta_3 + \sin 2\delta_4 \cos 2\delta_4)^2 = 0 \tag{69}$$

9 plies ($\delta_0 = 0$)

$$(3\sin 2\delta_2 + 2\sin 2\delta_3 + \sin 2\delta_4)^4 - 16(3\sin 2\delta_2 + 2\sin 2\delta_3 + \sin 2\delta_4)^2 + 16(3\sin 2\delta_2 \cos 2\delta_2 + 2\sin 2\delta_3 \cos 2\delta_3 + \sin 2\delta_4 \cos 2\delta_4)^2 = 0 \tag{70}$$





A second strategy is that of using uncoupled solutions of the quasi-trivial set

In such a set, antisymmetric sequences are sought for

For such sequences, $\mathbb{A}$ and $\mathbb{D}$ are orthotropic and in addition $\mathbb{B} = 0$

Some examples are given in the following table

Complete quasi-trivial stacking sequences for the uncoupled antisymmetric laminates up to 12 plies

		Ply number											
		1	2	3	4	5	6	7	8	9	10	11	12
7 plies	1	α	$-\alpha$	$-\alpha$	0	α	α	$-\alpha$	/	/	/	/	/
8 plies	2[a]	α	$-\alpha$	$-\alpha$	α	$-\alpha$	α	α	$-\alpha$	/	/	/	/
9 plies	3	α	$-\alpha$	$-\alpha$	α	0	$-\alpha$	α	α	$-\alpha$	/	/	/
	4[b]	α	$-\alpha$	0	$-\alpha$	0	α	0	α	$-\alpha$	/	/	/
	5[b]	0	α	$-\alpha$	$-\alpha$	0	α	α	$-\alpha$	0	/	/	/
10 plies	6	α	$-\alpha$	$-\alpha$	α	0	0	$-\alpha$	α	α	$-\alpha$	/	/
	7[a]	α	$-\alpha$	0	$-\alpha$	α	$-\alpha$	α	0	α	$-\alpha$	/	/
	8	α	0	$-\alpha$	$-\alpha$	$-\alpha$	α	α	α	0	$-\alpha$	/	/
	9	0	α	$-\alpha$	$-\alpha$	α	$-\alpha$	α	α	$-\alpha$	0	/	/
11 plies	10	α	$-\alpha$	$-\alpha$	α	0	0	0	$-\alpha$	α	α	$-\alpha$	/
	11	α	$-\alpha$	0	$-\alpha$	α	0	$-\alpha$	α	0	α	$-\alpha$	/
	12	α	$-\alpha$	0	0	$-\alpha$	0	α	0	0	α	$-\alpha$	/
	13	α	0	$-\alpha$	$-\alpha$	0	0	0	α	α	0	$-\alpha$	/
	14	0	α	$-\alpha$	$-\alpha$	α	0	$-\alpha$	α	α	$-\alpha$	0	/
	15	0	α	$-\alpha$	0	$-\alpha$	0	α	0	α	$-\alpha$	0	/
	16	0	0	α	$-\alpha$	$-\alpha$	0	α	α	$-\alpha$	0	0	/
12 plies	17	α	$-\alpha$	α	$-\alpha$	$-\alpha$	$-\alpha$	α	α	α	$-\alpha$	α	$-\alpha$
	18[b]	α	$-\alpha$	$-\alpha$	α	0	0	0	0	$-\alpha$	α	α	$-\alpha$
	19[b]	α	$-\alpha$	0	$-\alpha$	α	0	0	$-\alpha$	α	0	α	$-\alpha$
	20[c,b]	α	$-\alpha$	0	0	$-\alpha$	α	$-\alpha$	α	0	0	α	$-\alpha$
	21[c,b]	α	0	$-\alpha$	$-\alpha$	0	α	$-\alpha$	0	α	α	0	$-\alpha$
	22[b]	α	0	$-\alpha$	0	$-\alpha$	$-\alpha$	α	α	0	α	0	$-\alpha$
	23[c,b]	0	α	$-\alpha$	$-\alpha$	α	0	0	$-\alpha$	α	α	$-\alpha$	0
	24[b]	0	α	$-\alpha$	0	$-\alpha$	α	$-\alpha$	α	0	α	$-\alpha$	0
	25[b]	0	α	0	$-\alpha$	$-\alpha$	$-\alpha$	α	α	α	0	$-\alpha$	0
	26[b]	0	0	α	$-\alpha$	$-\alpha$	α	$-\alpha$	α	α	$-\alpha$	0	0

[a] Caprino and Crivelli-Visconti stacking sequences [2].
[b] Quasi-isotropic stacking sequences if all "0" are equal to 0° and "α" to 60° or −60°.
[c] Caprino and Crivelli-Visconti stacking sequences if all "0" are simultaneously equal to 0° or 90°, otherwise new solutions.

A third strategy is to use quasi-homogenous solutions of the quasi-trivial set

It is then sufficient to operate on $\mathbb{A}$ with the usual rules, to be applied to a quasi-trivial sequence, to get automatically uncoupling and fully orthotropy

So, if the laminate is balanced, or angle-ply, or quasi-isotropic, then $\mathbb{A}$ is orthotropic and because of quasi-homogeneity, $\mathbb{D} = \mathbb{A}$ and $\mathbb{B} = 0$

Here is some possible examples:

8 plies: [0 1 1 0 1 0 0 1]
12 plies: [0 1 0 1 1 1 0 0 0 1 0 1]
14 plies: [0 1 2 1 1 2 2 0 0 1 1 0 1 2]
16 plies: [0 1 2 2 0 2 0 1 1 2 0 2 0 0 1 2]
18 plies: [0 1 2 0 1 2 2 2 1 1 0 0 1 0 0 2 2 1]
20 plies: [0 1 2 1 2 2 1 2 1 2 0 1 1 0 2 0 2 1 1 2]
etc.

Isotropic laminates

In some cases, isotropy is sought for; this is namely the case when the forces do not have a precise direction but lightness is sought for using composite laminates

The search for isotropic laminates dates back to 1953, when Werren and Norris published a first study, giving a sufficient condition to obtain an isotropic tensor $\mathbb{A}$:

if the n plies of a laminate are subdivided into $g \geq 3$ groups, each one having the same number $\frac{n}{g}$ of layers, and if these groups have orientations that differ by an angle equal to $\frac{2\pi}{g}$, then the laminate will be isotropic in extension.

Possible solutions are $0°/60°/-60°$, $0°/45°/-45°/90°$, $0°/72°/144°/216°/288°$ etc.

This sufficient condition ensures only isotropy in extension, not in bending

It is normally used with symmetric stacks, in order to ensure also uncoupling

This is a serious drawback, because practically one needs to double the number of the plies

Another strategy, allowing also for finding fully isotropic laminates, is to apply the rule of Werren and Norris to quasi-homogeneous laminates of the quasi-trivial type (PV, GV, CompStruct, 2002)

As usual, in this case one gets a completely isotropic behavior and uncoupling

In this way, it has been possible to find fully isotropic laminates with the lowest number of plies: 5 unsymmetric 18-ply solutions

Some solutions are given in the following table

Number of plies	Orientations	Stacking sequence
18	$0 = -60°$	012012221100100221
	$1 = 0°$	012201120120201012
	$2 = 60°$	012201120210102021
		012210201102120021
		011222002100111220
24	$0 = -45°$	012323130201013120232301
	$1 = 0°$	
	$2 = 45°$	
	$3 = 90°$	
27	$0 = -60°$	001212122211000001220112120
	$1 = 0°$	001221211221000001210221210
	$2 = 60°$	010122021221210100102001212
		010122201221012100120001212
		010222101211012200120002121
30	$0 = 0°$	012330441244223113100002334421
	$1 = 72°$	012343021442413130200324101234
	$2 = 144°$	012344021334213120400432101243
	$3 = 216°$	012343021442431130200124303214
	$4 = 288°$	012344103224330211401430022143
		012343401224130241303410022134
		012344301224130231403410022143
		012344203131420432101320044213
		012344203314120132401420033241
		012334401224140331220013403142

Coming back to lamination parameters

Let us consider the case of orthotropic laminates in extension

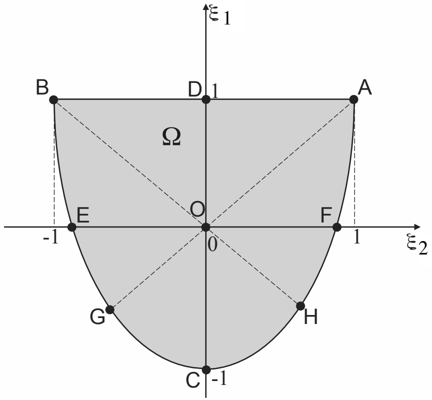
Then the only two non null lamination parameters are ξ_1 and ξ_2

If the laminate is an angle-ply $\pm\alpha$, then

$$\begin{aligned} \xi_1 &= \cos 4\alpha = 2\cos^2 2\alpha - 1 = 2\xi_2^2 - 1, \\ \xi_2 &= \cos 2\alpha, \quad -1 \leq \xi_2 \leq 1 \end{aligned} \tag{71}$$

So, in the plane $\xi_2 - \xi_1$, the angle-ply laminates are represented by an arc of parabola

It can be shown that this arc bounds the feasible domain of the lamination parameters Ω: all the laminates are represented by a lamination point, i.e. by a point of the plane $\xi_2 - \xi_1$, which is inside this domain, see the figure



- A$= (1, 1),\ \alpha = 0°$
- B$= (-1, 1),\ \alpha = 90°$
- C$= (0, -1),\ \alpha = \pm 45°$
- D$= (0, 1),\ \alpha = 0°, 90°$
- E$= (-\frac{1}{\sqrt{2}}, 0),\ \alpha = \pm 67.5°$
- F$= (\frac{1}{\sqrt{2}}, 0),\ \alpha = \pm 22.5°$
- G$= (-1/2, -1/2),\ \alpha = \pm 60°$
- H$= (1/2, -1/2),\ \alpha = \pm 30°$

The points indicated on the figure correspond to some special laminates

As said, the angle-ply laminates lie on the arc ACB

It is possible to put into relation lamination points on the contour with those inside the domain

In fact, let us consider balanced laminates; in this case, we can write

$$
\begin{aligned}
\xi_1 &= \sum_{k=1}^{n_g} \nu_k \cos 4\delta_k, \\
\xi_2 &= \sum_{k=1}^{n_g} \nu_k \cos 2\delta_k,
\end{aligned}
\tag{72}
$$

where

$$
\nu_k = \frac{n_k}{n} \tag{73}
$$

is the volume fraction of layers of the group k among the n_g groups of orientations.

The lamination parameters are hence linear functions of the volume fractions of the layers

So, the lamination parameters of the points of a segment linking two points of Ω represent laminates with volume fractions proportional to its distances from these two points

In other words, the vector of the volume fractions of a lamination point Q located between P and R, distant ℓ, is

$$\mathbf{v} = (1 - r)\mathbf{v}_P + r\mathbf{v}_R, \tag{74}$$

where $r\ell$ is the distance QP and $\mathbf{v}_P$ and $\mathbf{v}_R$ are the volume fraction vectors of P and R

So, points on the line AB represent cross-ply laminates

For instance, point P=(0.2,1) represent a cross-ply with 60% of plies at 0° and 40% at 90°; it is the case, e.g., of laminates $[0_3/90_2]_s$.

Nevertheless, any point of Ω belong to infinite lines

This means that there is not a bijective correspondence between lamination points, hence mechanical properties, and stacking sequences

The same properties can be obtained by different sequences: the solution is never unique in terms of the stacking sequence

For instance, the origin O corresponds to isotropic laminates ($\xi_1 = \xi_2 = 0$)

It can be obtained in different ways:

- segment AG: $OA=2/3\ell \rightarrow 2/3$ of the plies at $\pm 60^\circ$ and $1/3$ at 0°; so $\mathbb{A}$ is isotropic if an equal number of plies is distributed on the orientations $0^\circ, 60^\circ, -60^\circ$
- segment BH: same solution, rotated of 30°
- segment CD: O is at equal distance form C and D, hence an isotropic laminate is obtained putting an equal number of plies at $0^\circ, 45^\circ, -45^\circ, 90^\circ$
- segment EF: same solution, rotated of 22.5°

All these solutions are of the Werren and Norris type, but of course other solutions, more general, could be found

Evidently, these considerations apply not only to isotropic sequences, but to any other problem (for bending, it can be shown that the situation is absolutely identical)

The non uniqueness of the solution in terms of stacking sequence is not a problem, but really a fundamental aspect to tackle general problems of laminates design using a two-step modern approach, see the next lesson