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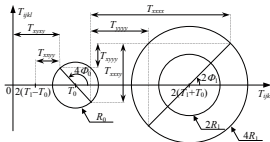
International Doctorate in Civil and Environmental Engineering

Anisotropic Structures - Theory and Design

Strutture anisotrope: teoria e progetto

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Topics of the fifth lesson

- The Polar Formalism - Part 2
- A short introduction to laminated anisotropic structures - Part 1

Recall of the polar formalism

$$T_{1111}(\theta) = T_0 + 2T_1 + R_0 \cos 4(\Phi_0 - \theta) + 4R_1 \cos 2(\Phi_1 - \theta)$$

$$T_{1112}(\theta) = R_0 \sin 4(\Phi_0 - \theta) + 2R_1 \sin 2(\Phi_1 - \theta)$$

$$T_{1122}(\theta) = -T_0 + 2T_1 - R_0 \cos 4(\Phi_0 - \theta)$$

$$T_{1212}(\theta) = T_0 - R_0 \cos 4(\Phi_0 - \theta)$$

$$T_{1222}(\theta) = -R_0 \sin 4(\Phi_0 - \theta) + 2R_1 \sin 2(\Phi_1 - \theta)$$

$$T_{2222}(\theta) = T_0 + 2T_1 + R_0 \cos 4(\Phi_0 - \theta) - 4R_1 \cos 2(\Phi_1 - \theta)$$

Some general remarks on elastic symmetries in \mathbb{R}^2

The results found in the previous Sections, deserve some commentary:

- from a purely **geometric** point of view, i.e. merely considering the elastic symmetries, **nothing differentiate ordinary orthotropy from the special orthotropy $R_0 = 0$** : both of them have only a couple of mutually orthogonal symmetry axes.
- From the **algebraic point of view, they are different**: they depend upon a different number of independent nonzero invariants and they are determined by invariant conditions concerning invariants of a different order.
- They also are interpreted differently: ordinary orthotropy corresponds to a precise value taken by the phase angle between the two anisotropic phases, R_0 -orthotropy to the vanishing of the anisotropic phase varying with 4θ .

- Also, while ordinary orthotropy preserves the same morphology also for the inverse tensor, though it is possible a change of type, from $K = 0$ to $k = 1$, R_0 -orthotropy does not preserve the same morphology for the compliance tensor, whose components depend upon the two anisotropic phases.
- From a **mechanical point of view**, R_0 -orthotropic materials have a behavior somewhat different from ordinary orthotropy, e.g. the components vary like those of a second-rank tensor or are isotropic.
- Square symmetric materials share some of the remarks done for R_0 -orthotropy, but geometrically speaking they are different from them and from ordinary orthotropy because they have two couples of mutually orthogonal symmetry axes tilted of $\pi/4$. This gives a periodicity of $\pi/2$ to all of the components.
- It can be seen that special orthotropies have some other interesting mechanical properties that are not possessed by ordinarily orthotropic materials.

All these remarks corroborate the idea that an **algebraic classification** of elastic symmetries, based upon the use of tensor invariants, is more effective than a purely geometric one.

In the end, there are **six possible cases of algebraically distinct elastic symmetries in \mathbb{R}^2** : ordinary orthotropy with $K = 0$ or $K = 1$, R_0 -orthotropy, r_0 -orthotropy, square symmetry and isotropy.

Table: Characteristics of the different types of elastic symmetries in \mathbb{R}^2 .

Symmetry type	Polar condition	Independent invariants	Nonzero invariants
Ordinary orthotropy $K = 0$	$\Phi_0 - \Phi_1 = 0$	4	$L_1, L_2, Q_1 = \left(\frac{C_1}{Q_2}\right)^2, Q_2, C_1$
Ordinary orthotropy $K = 1$	$\Phi_0 - \Phi_1 = \frac{\pi}{4}$	4	$L_1, L_2, Q_1 = \left(\frac{C_1}{Q_2}\right)^2, Q_2, C_1$
R_0 -orthotropy	$R_0 = 0$	3	L_1, L_2, Q_2
r_0 -orthotropy	$r_0 = 0$	3	L_1, L_2, Q_2
Square symmetry	$R_1 = 0$	3	L_1, L_2, Q_1
Isotropy	$R_0 = 0, R_1 = 0$	2	L_1, L_2

The polar formulae with the Kelvin's notation

All the relations given in the previous Sections for the polar formalism make use of the tensor notation, using four indexes.

We give here also their expression with the Kelvin's notation.

- Cartesian components

$$\begin{aligned}T_{11}(\theta) &= T_0 + 2T_1 + R_0 \cos 4(\Phi_0 - \theta) + 4R_1 \cos 2(\Phi_1 - \theta), \\T_{16}(\theta) &= \sqrt{2} [R_0 \sin 4(\Phi_0 - \theta) + 2R_1 \sin 2(\Phi_1 - \theta)], \\T_{12}(\theta) &= -T_0 + 2T_1 - R_0 \cos 4(\Phi_0 - \theta), \\T_{66}(\theta) &= 2 [T_0 - R_0 \cos 4(\Phi_0 - \theta)], \\T_{26}(\theta) &= \sqrt{2} [-R_0 \sin 4(\Phi_0 - \theta) + 2R_1 \sin 2(\Phi_1 - \theta)], \\T_{22}(\theta) &= T_0 + 2T_1 + R_0 \cos 4(\Phi_0 - \theta) - 4R_1 \cos 2(\Phi_1 - \theta).\end{aligned}\tag{1}$$

- Polar parameters

$$\begin{aligned}T_0 &= \frac{1}{8}(T_{11} - 2T_{12} + 2T_{66} + T_{22}), \\T_1 &= \frac{1}{8}(T_{11} + 2T_{12} + T_{22}), \\R_0 &= \frac{1}{8}\sqrt{(T_{11} - 2T_{12} - 2T_{66} + T_{22})^2 + 8(T_{16} - T_{26})^2}, \\R_1 &= \frac{1}{8}\sqrt{(T_{11} - T_{22})^2 + 2(T_{16} + T_{26})^2}, \\ \tan 4\Phi_0 &= \frac{2\sqrt{2}(T_{16} - T_{26})}{T_{11} - 2T_{12} - 2T_{66} + T_{22}}, \\ \tan 2\Phi_1 &= \frac{\sqrt{2}(T_{16} + T_{26})}{T_{11} - T_{22}}.\end{aligned}\tag{2}$$

Special planar anisotropic materials

The analysis of plane anisotropy made so far is tacitly based upon the assumption of **classical elastic body**.

The mechanical response of such a body is described by an elastic tensor \mathbb{E} having the minor and major symmetries.

Nevertheless, materials with different tensor symmetries can exist and we briefly consider them here:

- first we consider the so-called **rari-constant materials**, having supplementary tensor symmetries adding to the minor and major ones of classical materials
- then, we shortly analyze **complex materials**, calling with this name all the elastic materials that do not possess all of the minor or major symmetries

There is a characteristic fact in all these cases: **the number of tensor symmetries is linked to the number of tensor invariants**.

Rari-constant planar anisotropic materials

The idea of rari-constant materials stems from the early works of [Navier](#) and his model of matter, known as [molecular theory](#), first presented at *Académie des Sciences* on May 14, 1821.

Basically, the model proposed by Navier aims at explaining the behavior of elastic solids as that of a lattice of particles ([molecules](#)) interacting together via central forces proportional to their mutual distance.

This is not a new idea: it has its last foundation in the works of [Newton](#)

For what concerns the mechanics of solids, the true initiator of the molecular theory is considered to be [Boscovich](#)

Other works on this topic, before the *mémoire* of Navier, are those of [Poisson](#) on the equilibrium of bent plates, while subsequent fundamental contributions are due to [Cauchy](#) and [Saint-Venant](#)

The direct consequence of the molecular approach of Navier and Cauchy, the continuum as a limit of a discrete lattice of particles interacting together via central forces is that 15 moduli describe the behavior of a triclinic material, and only one modulus suffices for isotropy.

These results was not confirmed by experimental tests, so doubts existed about its validity, until the molecular approach was completely by-passed by the theory proposed in 1837 by [Green](#): matter is a continuum and the basic property defining the elastic behavior is [energetic](#): in non dissipative processes the internal forces derive from a quadratic potential.

This is the [multi-constant model](#): 21 independent moduli describe the elasticity of a triclinic body, and 2 that of an isotropic material.

The results of the Green's theory were confirmed by experience which, together with its much simpler theoretical background, ensured the success of the multi-constant theory.

Nonetheless, the diatribe between the molecular, rari-constant, and continuum, multi-constant, theories lasted a long period: **which is the right number of elastic constants and the correct model of elastic continuum?**

As an effect of this diatribe, the two models are usually considered as **opposing** and somewhat **irreconcilable**, though different researchers made attempts to show that this is not the case

The polar formalism applied to this problem has shed a new light on the matter, with some surprises (PV & BD, MMAS, 2015)

The only Cauchy-Poisson symmetry in \mathbb{R}^2 is

$$E_{1122} = E_{1212}. \quad (3)$$

Since now, we identify **rari-constant tensors** with those satisfying the Cauchy-Poisson condition

Identifying **rari-constant materials** is not so simple...

Theorem

\mathbb{E} is a rari-constant elastic tensor in $\mathbb{R}^2 \iff T_0 = T_1$.

Proof.

The proof is immediate: if \mathbb{E} is a rari-constant tensor, then

$$E_{1212}(\theta) = E_{1122}(\theta) \quad \forall \theta \Rightarrow$$

$$\begin{aligned} E_{1122}(\theta) &= -T_0 + 2T_1 - R_0 \cos 4(\Phi_0 - \theta), \\ E_{1212}(\theta) &= T_0 - R_0 \cos 4(\Phi_0 - \theta), \end{aligned} \quad \Rightarrow T_0 = T_1 \quad (4)$$

Conversely, if $T_0 = T_1$, then

$$\begin{aligned} 8T_0 &= E_{1111}(\theta) - 2E_{1122}(\theta) + 4E_{1212}(\theta) + E_{2222}(\theta) \\ 8T_1 &= E_{1111}(\theta) + 2E_{1122}(\theta) + E_{2222}(\theta) \quad \Rightarrow \end{aligned} \quad (5)$$

$$E_{1212}(\theta) = E_{1122}(\theta) \quad \forall \theta$$



- the number of independent tensor invariants is **linked to the number of index symmetries**; in particular, a supplementary index symmetry corresponds to the identity of two invariants, so that the number of independent invariants is decreased by one;
- only the isotropic part of \mathbb{E} can be multi-constant: the anisotropic part is not touched by the Cauchy-Poisson conditions, so that multi- and rari-constant materials **share all the same types of elastic symmetries**;
- the elastic bounds do not exclude the existence of the case $T_0 = T_1$: in the classical elastic setting, materials with a rari-constant tensor \mathbb{E} are **possible**;
- the existence of multi-constant materials with $T_0 = T_1$ **is not allowed**; this point is essential: because of Theorem 1 whenever $T_0 = T_1$, then tensor \mathbb{E} is **necessarily** rari-constant: $E_{1212}(\theta) = E_{1122}(\theta) \forall \theta$: a particular value of the tensor invariants determine a change of the algebraic structure of the elastic tensor;

A fundamental remark : all what has been said for \mathbb{E} is equally valid for \mathbb{S} \rightarrow : we can define a dual class of rari-constant materials, where the Cauchy-Poisson conditions are valid for the compliance tensor \mathbb{S} .

We name direct- and inverse- rari-constant materials those for which the Cauchy-Poisson condition holds respectively for \mathbb{E} or for \mathbb{S} .

These two classes are necessarily distinct, i.e. it cannot exist a material being at the same time direct- and inverse- rari-constant: the Cauchy-Poisson conditions cannot be satisfied at the same time by \mathbb{E} and \mathbb{S} .

That is why the name rari-constant has been used not only to denote a class of materials, but also a type of elastic tensor

Theorem

The Cauchy-Poisson condition (3) cannot be satisfied at the same time by \mathbb{E} and \mathbb{S} .

Proof.

Be \mathbb{E} rari-constant, i.e. $E_{1122} = E_{1212}$; then $T_0 = T_1$ by Theorem 1. The polar invariants of \mathbb{S} can then be calculated through eqs.

$$\begin{aligned}t_0 &= \frac{2}{\Delta} (T_0^2 - R_1^2), \\t_1 &= \frac{1}{2\Delta} (T_0^2 - R_0^2),\end{aligned}\tag{6}$$

It is then apparent that

$$t_0 = t_1 \iff T_0^2 = \frac{4R_1^2 - R_0^2}{3}.\tag{7}$$

This value of T_0 is incompatible with the elastic bounds and hence, $t_0 \neq t_1$ when $T_0 = T_1$, so by Theorem 1 applied to \mathbb{S} , $S_{1212} \neq S_{1122}$. □

The consequence is immediate: it is not correct to identify **automatically** rari-constant materials in \mathbb{R}^2 with the Cauchy-Poisson condition, because this concerns only one of the two elastic tensors of the material.

So, if \mathbb{E} is rari-constant, it has only 5 distinct components, but \mathbb{S} has 6 different components.

Conversely, if \mathbb{S} is rari-constant, it has 5 distinct components, but they are 6 for \mathbb{E} .

Nevertheless, in both the cases the number of independent tensor invariants is 4.

In fact, if \mathbb{E} is rari-constant, then $T_0 = T_1$ and by eqs. (6) we get

$$t_1 = \frac{T_0^2 - R_0^2}{4(T_0^2 - R_1^2)} t_0. \quad (8)$$

Hence, though $t_1 \neq t_0$, it is proportional to t_0 .

Of course, a similar relation exists for the dual case of \mathbb{S} .

Direct rari-constant materials

$$\begin{aligned}E_{1212}(\theta) &= E_{1122}(\theta) \quad \forall \theta, \\T_0 &= T_1, \\S_{1212}(\theta) &\neq S_{1122}(\theta), \\t_1 &= \frac{T_0^2 - R_0^2}{4(T_0^2 - R_1^2)} t_0,\end{aligned}\tag{9}$$

and

$$\begin{aligned}E_{1111}(\theta) &= 3T_0 + R_0 \cos 4(\Phi_0 - \theta) + 4R_1 \cos 2(\Phi_1 - \theta), \\E_{1112}(\theta) &= R_0 \sin 4(\Phi_0 - \theta) + 2R_1 \sin 2(\Phi_1 - \theta), \\E_{1122}(\theta) &= E_{1212}(\theta) = T_0 - R_0 \cos 4(\Phi_0 - \theta), \\E_{1222}(\theta) &= -R_0 \sin 4(\Phi_0 - \theta) + 2R_1 \sin 2(\Phi_1 - \theta), \\E_{2222}(\theta) &= 3T_0 + R_0 \cos 4(\Phi_0 - \theta) - 4R_1 \cos 2(\Phi_1 - \theta).\end{aligned}\tag{10}$$

Inverse rari-constant materials

$$\begin{aligned}S_{1212}(\theta) &= S_{1122}(\theta) \quad \forall \theta, \\t_0 &= t_1, \\E_{1212}(\theta) &\neq E_{1122}(\theta) \quad \forall \theta, \\T_1 &= \frac{t_0^2 - r_0^2}{4(t_0^2 - r_1^2)} T_0,\end{aligned}\tag{11}$$

and

$$\begin{aligned}S_{1111}(\theta) &= 3t_0 + r_0 \cos 4(\varphi_0 - \theta) + 4r_1 \cos 2(\varphi_1 - \theta), \\S_{1112}(\theta) &= r_0 \sin 4(\varphi_0 - \theta) + 2r_1 \sin 2(\varphi_1 - \theta), \\S_{1122}(\theta) &= S_{1212}(\theta) = t_0 - r_0 \cos 4(\varphi_0 - \theta), \\S_{1222}(\theta) &= -r_0 \sin 4(\varphi_0 - \theta) + 2r_1 \sin 2(\varphi_1 - \theta), \\S_{2222}(\theta) &= 3t_0 + r_0 \cos 4(\varphi_0 - \theta) - 4r_1 \cos 2(\varphi_1 - \theta).\end{aligned}\tag{12}$$

Finally, the only necessary and sufficient condition for identifying a rari-constant material, regardless of its type, i.e. independently of the number of distinct Cartesian components for \mathbb{E} or \mathbb{S} , is that the number of **independent linear tensor invariants must be one**.

It is actually possible to fabricate both the cases of direct- and inverse- rari-constant layers.

This can be done using appropriate volume fractions of unidirectional fibers to reinforce an isotropic matrix.

If we use the classical technical laws of homogenization, we finally get the following conditions to be satisfied

$(E_f = m E_m, \nu_f = n \nu_m, \alpha = \frac{t_0^2 - r_0^2}{4(t_0^2 - r_1^2)}, \nu_f: \text{volume fraction})$

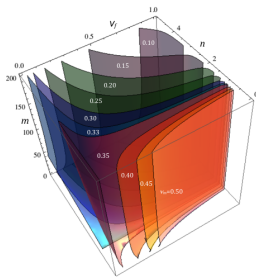
- direct rari-constant materials:

$$\begin{aligned} & [1 + (m - 1)\nu_f][m + \nu_f(1 - m)] - m \nu_m^2 [1 + (n - 1)\nu_f]^2 - \\ & 2\nu_m [1 + (m - 1)\nu_f][1 + (n - 1)\nu_f][m(1 - \nu_f)(1 + \nu_m) + \nu_f(1 + n \nu_m)] = 0 \end{aligned} \quad (13)$$

- inverse rari-constant materials:

$$\begin{aligned}
 & [1 + (m - 1)v_f] \{ [(m - 1)^2 v_f^2 - (m - 1)^2 v_f - 2m](\alpha - 1) + \\
 & 2m(\alpha + 1)[1 + v_f(n - 1)]\nu_m \} [v_f(1 + n\nu_m) + m(1 - v_f)(1 + \nu_m)] + \\
 & 2m\alpha \{ (v_f - 1)v_f + m^2(v_f - 1)v_f + \\
 & m[[1 + v_f(n - 1)]^2 \nu_m^2] - 2v_f^2 + 2v_f - 1 \} = 0
 \end{aligned} \tag{14}$$

Direct rari-constant materials



Inverse rari-constant materials

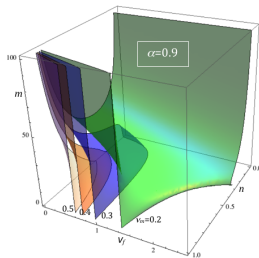


Figure: Solutions for rari-constant anisotropic layers.

Complex anisotropic layers

The use of the polar formalism allowed to study the **complexity** of the anisotropy of **complex materials** (PV & GV, IJSS, 2010)

First case: an elastic tensor without the minor symmetries: 10 independent components, 9 invariants:

$$\begin{aligned}E_{1111} &= T_0 + T_1 + T_2 + R_0 \cos 4\Phi_0 + 2R_1 \cos 2\Phi_1 + 2R_2 \cos 2\Phi_2, \\E_{1112} &= -T_3 + R_0 \sin 4\Phi_0 + 2R_2 \sin 2\Phi_2, \\E_{1121} &= T_3 + R_0 \sin 4\Phi_0 + 2R_1 \sin 2\Phi_1, \\E_{1122} &= -T_0 + T_1 + T_2 - R_0 \cos 4\Phi_0, \\E_{1212} &= T_0 + T_1 - T_2 - R_0 \cos 4\Phi_0 + 2R_1 \cos 2\Phi_1 - 2R_2 \cos 2\Phi_2, \\E_{1221} &= T_0 - T_1 + T_2 - R_0 \cos 4\Phi_0, \\E_{1222} &= -T_3 - R_0 \sin 4\Phi_0 + 2R_1 \sin 2\Phi_1, \\E_{2121} &= T_0 + T_1 - T_2 - R_0 \cos 4\Phi_0 - 2R_1 \cos 2\Phi_1 + 2R_2 \cos 2\Phi_2, \\E_{2112} &= T_3 - R_0 \sin 4\Phi_0 + 2R_2 \sin 2\Phi_2, \\E_{2222} &= T_0 + T_1 + T_2 + R_0 \cos 4\Phi_0 - 2R_1 \cos 2\Phi_1 - 2R_2 \cos 2\Phi_2.\end{aligned}\tag{15}$$

Invariants: all the polar moduli T_0, T_1 etc. and the differences of the polar angles $\Phi_0 - \Phi_2$ and $\Phi_1 - \Phi_2$.

Ordinary orthotropy

$$\Phi_0 - \Phi_1 = K_{01} \frac{\pi}{4}, \quad \Phi_0 - \Phi_2 = K_{02} \frac{\pi}{4}, \quad \Phi_1 - \Phi_2 = K_{12} \frac{\pi}{2}. \quad (16)$$

→ 4 possible different ordinary orthotropies :

$K_{02} = K_{12} = 0$, $K_{02} = 1$ and $K_{12} = 0$, $K_{02} = 0$ and $K_{12} = 1$,
 $K_{02} = K_{12} = 1$.

6 special orthotropies

$$\begin{aligned} R_0 = 0, K_{12} = 0, & \quad R_0 = 0, K_{12} = 1, \\ R_1 = 0, K_{02} = 0, & \quad R_1 = 0, K_{02} = 1, \\ R_2 = 0, K_{01} = 0, & \quad R_2 = 0, K_{01} = 1. \end{aligned} \quad (17)$$

Isotropy :

$$T_3 = R_0 = R_1 = R_2 = 0; \quad (18)$$

The condition on T_3 ensures the invariance of the material response under a mirror symmetry about an axis.

The relations between the Cartesian and polar components in this case are

$$\begin{aligned} E_{1111} &= E_{2222} = T_0 + T_1 + T_2, \\ E_{1122} &= -T_0 + T_1 + T_2, \\ E_{1212} &= E_{2121} = T_0 + T_1 - T_2, \\ E_{1221} &= T_0 - T_1 + T_2, \end{aligned} \quad (19)$$

the remaining components being null.

Isotropy is hence determined by **three independent moduli**, not by two as for classical materials.

Second case: an elastic tensor without the major symmetries: 9 independent components, 8 invariants:

$$\begin{aligned}E_{1111} &= T_0 + 2T_1 + R_0 \cos 4\Phi_0 + 2R_1 \cos 2\Phi_1 + 2R_2 \cos 2\Phi_2, \\E_{1112} &= -T_3 + R_0 \sin 4\Phi_0 + 2R_2 \sin 2\Phi_2, \\E_{1122} &= -T_0 + 2T_1 - R_0 \cos 4\Phi_0 + 2R_1 \cos 2\Phi_1 - 2R_2 \cos 2\Phi_2, \\E_{1211} &= T_3 + R_0 \sin 4\Phi_0 + 2R_1 \sin 2\Phi_1, \\E_{1212} &= T_0 - R_0 \cos 4\Phi_0, \\E_{1222} &= -T_3 - R_0 \sin 4\Phi_0 + 2R_1 \sin 2\Phi_1, \\E_{2211} &= -T_0 + 2T_1 - R_0 \cos 4\Phi_0 - 2R_1 \cos 2\Phi_1 + 2R_2 \cos 2\Phi_2, \\E_{1121} &= T_3 - R_0 \sin 4\Phi_0 + 2R_2 \sin 2\Phi_2, \\E_{2222} &= T_0 + 2T_1 + R_0 \cos 4\Phi_0 - 2R_1 \cos 2\Phi_1 - 2R_2 \cos 2\Phi_2.\end{aligned}\tag{20}$$

The only difference is the lacking of $T_3 \rightarrow$ the anisotropy does not change with respect to the previous case and isotropy perfectly coincides with that of classical materials

These examples clearly show that there is an **influence of the tensor symmetries, i.e. of the algebraic structure of the elastic tensor, on the elastic symmetries**

Invariant formulation of anisotropic strength criteria

Stress tensor criteria for anisotropic layers:

- Tsai-Hill criterion:

$$F_{Hill} = \{\sigma\}^T [F] \{\sigma\} \leq 1, \quad [F] = \begin{bmatrix} \frac{1}{X^2} & -\frac{1}{2X^2} & 0 \\ -\frac{1}{2X^2} & \frac{1}{Y^2} & 0 \\ 0 & 0 & \frac{1}{S^2} \end{bmatrix}. \quad (21)$$

- Hoffman criterion:

$$F_{Hoff} = \{\sigma\}^T [F] \{\sigma\} + \{\sigma\}^T \{f\} \leq 1, \quad (22)$$

where

$$[F] = \begin{bmatrix} \frac{1}{X_t X_c} & -\frac{1}{2X_t X_c} & 0 \\ -\frac{1}{2X_t X_c} & \frac{1}{Y_t Y_c} & 0 \\ 0 & 0 & \frac{1}{S^2} \end{bmatrix}, \quad \{f\} = \begin{Bmatrix} \frac{X_c - X_t}{X_t X_c} \\ \frac{Y_c - Y_t}{Y_t Y_c} \\ 0 \end{Bmatrix}. \quad (23)$$

- Tsai-Wu criterion:

$$F_{TW} = \{\sigma\}^T [F] \{\sigma\} + \{\sigma\}^T \{f\} \leq 1, \quad (24)$$

where, for orthotropic layers it is

$$[F] = \begin{bmatrix} \frac{1}{X_t X_c} & \frac{F_{12}^*}{\sqrt{X_t X_c Y_t Y_c}} & 0 \\ \frac{F_{12}^*}{\sqrt{X_t X_c Y_t Y_c}} & \frac{1}{Y_t Y_c} & 0 \\ 0 & 0 & \frac{1}{S^2} \end{bmatrix}, \quad \{f\} = \begin{Bmatrix} \frac{X_c - X_t}{X_t X_c} \\ \frac{Y_c - Y_t}{Y_t Y_c} \\ 0 \end{Bmatrix}. \quad (25)$$

- It is possible to give a unified matrix formulation to these criteria:

$$F_{..} = \{\sigma\}^T [F] \{\sigma\} + \{\sigma\}^T \{f\} \leq 1, \quad (26)$$

with

$$[F] = \begin{bmatrix} F_{xx} & F_{xy} & 0 \\ F_{xy} & F_{yy} & 0 \\ 0 & 0 & F_{ss} \end{bmatrix}, \quad \{f\} = \begin{Bmatrix} f_x \\ f_y \\ 0 \end{Bmatrix}. \quad (27)$$

	Tsai-Hill	Hoffman	Tsai-Wu
F_{xx}	$\frac{1}{Y^2}$	$\frac{1}{Y_t Y_c}$	$\frac{1}{Y_t Y_c}$
F_{xy}	$-\frac{1}{2X^2}$	$-\frac{1}{2X_t X_c}$	$\frac{F_{12}^*}{\sqrt{X_t X_c Y_t Y_c}}$
F_{yy}	$\frac{1}{X^2}$	$\frac{1}{X_t X_c}$	$\frac{1}{X_t X_c}$
F_{ss}	$\frac{1}{S^2}$	$\frac{1}{S^2}$	$\frac{1}{S^2}$

Table: Terms of $[F]$ for the three criteria.

	Tsai-Hill	Hoffman	Tsai-Wu
f_x	0	$\frac{Y_c - Y_t}{Y_t Y_c}$	$\frac{Y_c - Y_t}{Y_t Y_c}$
f_y	0	$\frac{X_c - X_t}{X_t X_c}$	$\frac{X_c - X_t}{X_t X_c}$
f_s	0	0	0

Table: Terms of $\{f\}$ for the three criteria.

Polar formulation of the strength criteria:

- We first write the unified formulation of strength criteria in tensor form

$$F_{\dots} = \boldsymbol{\sigma} \cdot \mathbb{F} \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \mathbf{f} \leq 1, \quad (28)$$

- Then, as tacitly done in all the criteria, we assume that:
 - \mathbb{F} is an elasticity-like tensor
 - \mathbf{f} is a symmetric 2nd-rank tensor
 - both of them are **weakness tensors**
- We can then represent \mathbb{F} and \mathbf{f} by their polar components:
 - be $\gamma_0, \gamma_1, \lambda_0, \lambda_1, \omega_0, \omega_1$ the polar components of \mathbb{F} and
 - γ, λ, ω those of \mathbf{f} .
- Then:

$$\left\{ \begin{array}{l} F_{xxxx} = \gamma_0 + 2\gamma_1 + \lambda_0 \cos 4\omega_0 + 4\lambda_1 \cos 2\omega_1, \\ F_{xxyy} = \gamma_0 - 2\gamma_1 - \lambda_0 \cos 4\omega_0, \\ F_{xyxy} = \gamma_0 - \lambda_0 \cos 4\omega_0, \\ F_{yyxy} = -\lambda_0 \sin 4\omega_0 + 2\lambda_1 \sin 2\omega_1, \\ F_{yyyy} = \gamma_0 + 2\gamma_1 + \lambda_0 \cos 4\omega_0 - 4\lambda_1 \cos 2\omega_1. \end{array} \right. \quad (29)$$

$$\left\{ \begin{array}{l} f_{xx} = \gamma + \lambda \cos 2\omega, \\ f_{yy} = \gamma - \lambda \cos 2\omega, \\ f_{xy} = \lambda \sin 2\omega. \end{array} \right. \quad (30)$$

Relation between $[F]$, $\{f\}$ and \mathbb{F} , \mathbf{f} :

$$\left\{ \begin{array}{l} F_{xx} = F_{xxxx}, \\ F_{xs} = 2F_{xxyy}, \\ F_{xy} = F_{xyxy}, \\ F_{ss} = 4F_{xyxy}, \\ F_{ys} = 2F_{xyxy}, \\ F_{yy} = F_{yyyy}; \end{array} \right. \quad \left\{ \begin{array}{l} f_x = f_{xx}, \\ f_y = f_{yy}, \\ f_s = 2f_{xy}. \end{array} \right. \quad (31)$$

Inversely,

$$\left\{ \begin{array}{l} 8\gamma_0 = F_{xxxx} - 2F_{xxyy} + 4F_{xyxy} + F_{yyyy}, \\ 8\gamma_1 = F_{xxxx} + 2F_{xxyy} + F_{yyyy}, \\ 8\lambda_0 e^{4i\omega_0} = F_{xxxx} + 4iF_{xoxy} - 2F_{xxyy} - 4F_{xyxy} - 4iF_{xyyy} + F_{yyyy}, \\ 8\lambda_1 e^{2i\omega_1} = F_{xxxx} + 2iF_{xoxy} + 2iF_{xyyy} - F_{yyyy}, \end{array} \right. \quad (32)$$

and

$$\left\{ \begin{array}{l} \gamma = \frac{f_{xx} + f_{yy}}{2}, \\ \lambda e^{2i\omega} = \frac{f_{xx} - f_{yy}}{2} + if_{xy}. \end{array} \right. \quad (33)$$

In terms of strength properties, for an **orthotropic layer** ($\omega_0 - \omega_1 = l \frac{\pi}{4}$, $l = 0, 1$), the polar components of \mathbb{F} are

$$\left\{ \begin{array}{l} 8\gamma_0 = \frac{1}{X_t X_c} + \frac{1}{Y_t Y_c} + \frac{1}{S^2} - 2F_{xxyy}, \\ 8\gamma_1 = \frac{1}{X_t X_c} + \frac{1}{Y_t Y_c} + 2F_{xxyy}, \\ 8(-1)^l \lambda_0 e^{4i\omega_1} = \frac{1}{X_t X_c} + \frac{1}{Y_t Y_c} - \frac{1}{S^2} - 2F_{xxyy}, \\ 8\lambda_1 e^{2i\omega_1} = \frac{1}{Y_t Y_c} - \frac{1}{X_t X_c}. \end{array} \right. \quad (34)$$

Because $\lambda_0, \lambda_1 > 0$, then

$$\begin{aligned} \frac{2}{X_t X_c} + \frac{1}{Y_t Y_c} - 2F_{xxyy} &> \frac{1}{S^2} && \text{if } l = 0, \\ \frac{2}{X_t X_c} + \frac{1}{Y_t Y_c} - 2F_{xxyy} &< \frac{1}{S^2} && \text{if } l = 1, \end{aligned} \quad (35)$$

$$Y_t Y_c < X_t X_c .$$

In the same way, for tensor \mathbf{f} we get

$$\left\{ \begin{array}{l} 2\gamma = \frac{Y_c - Y_t}{Y_t Y_c} + \frac{X_c - X_t}{X_t X_c}, \\ 2\lambda e^{2i\omega} = \frac{Y_c - Y_t}{Y_t Y_c} - \frac{X_c - X_t}{X_t X_c}. \end{array} \right. \quad (36)$$

Concerning Tsai-Wu's criterion, some polar parameters are **linear function of the term** $F_{12}^* \in [-1; 1]$:

$$\left\{ \begin{array}{l} 8\gamma_0 = \frac{1}{X_t X_c} + \frac{1}{Y_t Y_c} + \frac{1}{S^2} - 2 \frac{F_{12}^*}{\sqrt{X_t X_c Y_t Y_c}}, \\ 8\gamma_1 = \frac{1}{X_t X_c} + \frac{1}{Y_t Y_c} + 2 \frac{F_{12}^*}{\sqrt{X_t X_c Y_t Y_c}}, \\ 8(-1)^l \lambda_0 e^{4i\omega_1} = \frac{1}{X_t X_c} + \frac{1}{Y_t Y_c} - \frac{1}{S^2} - 2 \frac{F_{12}^*}{\sqrt{X_t X_c Y_t Y_c}}. \end{array} \right. \quad (37)$$

A value of F_{12}^* **cannot be fixed** because it depends upon the experimental test, i.e. on the stress field.

Nevertheless, thanks to the polar formalism, we can see that the term F_{12}^* is independent from the stress field in 2 cases:

$$\lambda_0 = 0 : F_{12}^* = \frac{\sqrt{X_t X_c Y_t Y_c}}{2} \left(\frac{1}{X_t X_c} + \frac{1}{Y_t Y_c} - \frac{1}{S^2} \right), \quad (38)$$

$$\text{Isotropy, i.e. } \lambda_0 = \lambda_1 = 0 : F_{12}^* = 1 - \frac{X_t X_c}{2S^2}.$$

Let us give the example of two materials:

Material	X_t	X_c	Y_t	Y_c	S
E-Glass Epoxy	1080	620	39	128	89
Carbon/Epoxy	1447	1447	51.7	206	93

Table: Mechanical strength properties of orthotropic materials, [MPa].

The corresponding values of the polar components are:

		$\times 10^{-5}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$		
X	Y	$\gamma_0[\text{MPa}^{-2}]$	$\gamma_1[\text{MPa}^{-2}]$	$\lambda_0[\text{MPa}^{-2}]$	$\lambda_1[\text{MPa}^{-2}]$	l	ω_1
X_T	Y_T	9.807	82.29	66.51	82.08	0	0
X_C	Y_C	2.373	7.955	7.826	7.304	1	0
X_T	Y_C	2.352	7.737	8.044	7.522	1	0
X_C	Y_T	9.829	82.51	66.73	81.86	0	0

Table: Weakness Polar Components for the generic E-Glass Epoxy in Tsai-Hill f.c.

	$\times 10^{-5}$	$\times 10^{-5}$	$\times 10^{-6}$	$\times 10^{-5}$			$\times 10^{-3}$	$\times 10^{-3}$	
Material	$\gamma_0[\text{MPa}^{-2}]$	$\gamma_1[\text{MPa}^{-2}]$	$\lambda_0[\text{MPa}^{-2}]$	$\lambda_1[\text{MPa}^{-2}]$	l	ω_1	$\gamma[\text{MPa}^{-1}]$	$\lambda[\text{MPa}^{-1}]$	ω
E-Glass Epoxy	4.12	2.50	9.63	2.49	0	0	8.57	9.26	0
Carbon/Epoxy	2.63	1.17	2.60	1.17	1	0	7.24	7.24	0

Table: Weakness Polar Components for orthotropic materials in Hoffman f.c.

	$\times 10^{-5}$	$\times 10^{-5}$	$\times 10^{-6}$	$\times 10^{-5}$			$\times 10^{-3}$	$\times 10^{-3}$	
Material	$\gamma_0[\text{MPa}^{-2}]$	$\gamma_1[\text{MPa}^{-2}]$	$\lambda_0[\text{MPa}^{-2}]$	$\lambda_1[\text{MPa}^{-2}]$	ν	ω_1	$\gamma[\text{MPa}^{-1}]$	$\lambda[\text{MPa}^{-1}]$	ω
E-Glass Epoxy	[4.53:3.67]	[2.09:2.96]	[13.8:5.12]	2.49	0	0	8.57	9.26	0
Carbon/Epoxy	[2.79:2.46]	[1.01:1.35]	[0.98:4.33]	1.17	1	0	7.24	7.24	0

Table: Weakness Polar Components for orthotropic materials in Tsai-Wu f.c.

	$\times 10^{-5}$	$\times 10^{-5}$	$\times 10^{-6}$	$\times 10^{-5}$			$\times 10^{-3}$	$\times 10^{-3}$	
Material	$\gamma_0[\text{MPa}^{-2}]$	$\gamma_1[\text{MPa}^{-2}]$	$\lambda_0[\text{MPa}^{-2}]$	$\lambda_1[\text{MPa}^{-2}]$	ν	ω_1	$\gamma[\text{MPa}^{-1}]$	$\lambda[\text{MPa}^{-1}]$	ω
E-Glass Epoxy	[4.53:3.67]	[2.09:2.96]	[13.8:5.12]	2.49	0	0	8.57	9.26	0
Carbon/Epoxy	[2.79:2.46]	[1.01:1.35]	[0.98:4.33]	1.17	1	0	7.24	7.24	0

Table: Weakness Polar Components for orthotropic materials in Tsai-Wu f.c.

We can now give the invariant polar expression of the failure criteria (T, R, Φ : polar parameters of σ):

- Tsai-Hill failure criterion:

$$F_{Hill} = 4R^2\gamma_0 + 8T^2\gamma_1 + (-1)^l 4\lambda_0 R^2 \cos 4(\omega_1 - \Phi) + 16TR\lambda_1 \cos 2(\omega_1 - \Phi) \leq 1. \quad (39)$$

- Hoffman and Tsai-Wu failure criteria:

$$F_{Hoff/TW} = 4R^2\gamma_0 + 8T^2\gamma_1 + (-1)^l 4\lambda_0 R^2 \cos 4(\omega_1 - \Phi) + 16TR\lambda_1 \cos 2(\omega_1 - \Phi) + 2T\gamma + 2\lambda R \cos 2(\omega - \Phi) \leq 1. \quad (40)$$

Anisotropic damage of isotropic layers

Problem: an initially isotropic layer, when stressed can be damaged and become anisotropic; how much?

Damage model (Chaboche 1979, Leckie and Onat 1980, Sidoroff 1980, Chow 1987):

\mathbb{Q} : initial stiffness, $\tilde{\mathbb{Q}}$: damaged stiffness, $\hat{\mathbb{Q}}$: loss of stiffness

$$\tilde{\mathbb{Q}} = [(\mathbb{I} - \mathbb{D})\mathbb{Q}]^{Sym} \Rightarrow \tilde{\mathbb{Q}} = \mathbb{Q} - \hat{\mathbb{Q}} \quad \text{with} \quad \hat{\mathbb{Q}} = \frac{\mathbb{Q}\mathbb{D} + \mathbb{D}\mathbb{Q}}{2},$$

- \mathbb{Q} and $\tilde{\mathbb{Q}}$ are positive definite, while \mathbb{D} and $\hat{\mathbb{Q}}$ are semi definite

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- \mathbb{Q} and $\tilde{\mathbb{Q}}$ are positive definite, while \mathbb{D} and $\hat{\mathbb{Q}}$ are semi definite
- imposing the positive semi definiteness of $\hat{\mathbb{Q}}$ and the positive definiteness of $\tilde{\mathbb{Q}}$, gives the bounds on \mathbb{D} and $\hat{\mathbb{Q}}$ (the positive semi definiteness of $\hat{\mathbb{Q}}$ \Rightarrow that of \mathbb{D})

Advantages of the polar formalism:

- the polar bounds on a fourth-rank tensor concern invariant quantities and are valid for any type of anisotropy \Rightarrow any type of anisotropic damage can be investigated
- each one of the polar parameters of $\tilde{\mathbb{Q}}$ depends exclusively upon the corresponding polar parameter of \mathbb{D} \Rightarrow the polar formalism allows for uncoupling the expressions of the parameters of $\tilde{\mathbb{Q}}$ as functions of those of \mathbb{D}

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- each one of the polar parameters of $\tilde{\mathbb{Q}}$ depends exclusively upon the corresponding polar parameter of \mathbb{D} \Rightarrow the polar formalism allows for uncoupling the expressions of the parameters of $\tilde{\mathbb{Q}}$ as functions of those of \mathbb{D}

$$\begin{aligned}\tilde{T}_0 &= T_0(1 - 2D_0), & \tilde{T}_1 &= T_1(1 - 4D_1), \\ \tilde{R}_0 &= 2T_0S_0, & \tilde{R}_1 &= (T_0 + 2T_1)S_1, \\ \tilde{\Phi}_0 &= \Psi_0 + \frac{\pi}{4}, & \tilde{\Phi}_1 &= \Psi_1 + \frac{\pi}{2}.\end{aligned}$$

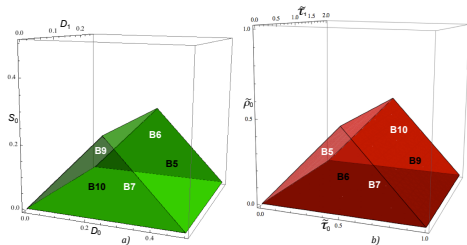
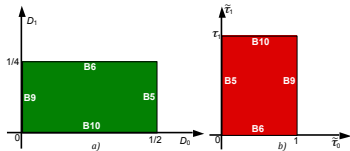
$D_0, D_1, S_0, S_1, \Psi_0, \Psi_1$: polar parameters of \mathbb{D}

$\tilde{T}_0, \tilde{T}_1, \tilde{R}_0, \tilde{R}_1, \tilde{\Phi}_0, \tilde{\Phi}_1$: polar parameters of $\tilde{\mathbb{Q}}$

Table: Minimal set of polar bounds in the completely anisotropic case

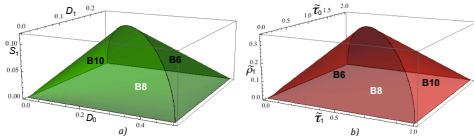
$$(\tau_1 = \frac{2T_1}{T_0}, \tilde{\tau}_0 = \frac{\tilde{T}_0}{T_0}, \tilde{\tau}_1 = \frac{2\tilde{T}_1}{T_0}, \tilde{\rho}_0 = \frac{\tilde{R}_0}{T_0}, \tilde{\rho}_1 = \frac{\tilde{R}_1}{T_0})$$

	Polar bounds for \mathbb{D}	Polar bounds for $\tilde{\mathbb{Q}}$
B5	$2(D_0 + S_0) < 1$	$\tilde{\tau}_0 > \tilde{\rho}_0$
B6	$\frac{\tau_1}{4(1+\tau_1)^2} (1 - 4D_1)[(1 - 2D_0)^2 - 4S_0^2] > S_1^2[1 - 2D_0 + 2S_0 \cos 4(\Psi_0 - \Psi_1)]$	$\tilde{\tau}_1(\tilde{\tau}_0^2 - \tilde{\rho}_0^2) > 4\tilde{\rho}_1^2 [\tilde{\tau}_0 - \tilde{\rho}_0 \cos 4(\tilde{\Phi}_0 - \tilde{\Phi}_1)]$
B7	$S_0 \geq 0$	$\tilde{\rho}_0 \geq 0$
B8	$S_1 \geq 0$	$\tilde{\rho}_1 \geq 0$
B9	$D_0 \geq S_0$	$\tilde{\tau}_0 + \tilde{\rho}_0 \leq 1$
B10	$D_1(D_0^2 - S_0^2) \geq \frac{(1+\tau_1)^2}{2\tau_1} S_1^2 [D_0 - S_0 \cos 4(\Psi_0 - \Psi_1)]$	$(\tau_1 - \tilde{\tau}_1)[(1 - \tilde{\tau}_0)^2 - \tilde{\rho}_0^2] \geq 4\tilde{\rho}_1^2 [1 - \tilde{\tau}_0 + \tilde{\rho}_0 \cos 4(\tilde{\Phi}_0 - \tilde{\Phi}_1)]$

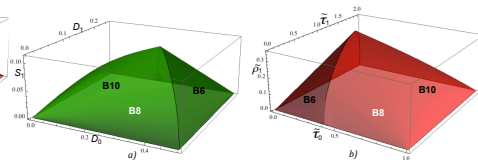


$\tilde{\mathbb{Q}}$ isotropic

$\tilde{\mathbb{Q}}$ R_1 -orthotropic (square symmetric)



$\tilde{\mathbb{Q}}$ R_0 -orthotropic



$\tilde{\mathbb{Q}}$ r_0 -orthotropic

Some examples of planar anisotropic materials

Table 4.9: Examples of planar anisotropic materials.

	Wood ¹	Carbon/Epoxy ²	Boron/Epoxy ³	S-Glass/Epoxy ⁴	Kevlar/Epoxy ⁵	Ice ⁶	TiB ₂ ⁷	Carbon/Epoxy ⁸	Glass/Epoxy ⁹	BR45a ¹⁰	BR60 ¹¹
E_1	10.04	181.00	205.00	47.66	86.90	11.75	253.81	54.00	29.70	40.40	30.90
E_2	0.42	10.30	18.50	13.31	5.52	9.61	387.60	54.00	29.70	19.60	42.60
G_{12}	0.75	7.17	5.59	4.75	2.14	3.00	250.00	4.00	5.30	25.00	14.00
ν_{12}	0.23	0.28	0.23	0.27	0.34	0.27	0.29	0.04	0.17	0.75	0.34
E_{max}	10.04	181.00	205.00	47.66	86.90	11.75	510.51	54.00	29.70	57.82	42.60
$\theta_{E_{max}}(^{\circ})$	0	0	0	0	0	0	53.19	0.90	0.90	34.44	90
E_{min}	0.42	10.30	15.62	12.36	5.26	8.33	253.81	14.02	16.35	19.60	30.90
$\theta_{E_{min}}(^{\circ})$	90	90	57.35	60.8	64.73	50.60	0	45	45	90	0
Q_{11}	10.06	181.81	206.00	48.65	87.54	12.51	291.76	54.11	30.58	55.56	36.76
Q_{22}	0.42	10.35	18.59	13.59	5.56	10.22	445.56	54.11	30.58	26.96	50.68
Q_{66}	1.50	14.34	11.18	9.50	4.28	6.00	500.00	8.00	10.60	50.00	28.00
Q_{12}	0.10	2.89	4.27	3.67	1.89	2.78	130.05	2.43	5.20	20.22	17.23
T_0	1.66	26.88	29.80	92.38	12.23	3.65	184.65	14.92	8.99	17.76	13.62
T_1	1.34	24.74	29.14	86.97	12.11	3.54	124.71	14.14	8.94	15.37	15.24
R_0	0.91	19.71	24.21	44.86	10.09	0.65	65.35	10.92	3.70	7.24	0.38
R_1	1.20	21.43	23.42	43.82	10.25	0.28	19.23	0	0	3.57	1.74
Φ_0	0	0	0	0	0	0	$\pi/4$	0	0	$\pi/4$	$\pi/4$
Φ_1	0	0	0	0	0	0	$\pi/2$	0	0	0	$\pi/2$
K	0	0	0	0	0	0	1	-	-	1	1
ρ	0.76	0.92	1.03	1.02	0.98	2.32	3.40	∞	∞	2.03	0.22
τ	2.88	2.62	2.61	4.25	2.53	15.16	6.37	3.96	7.26	6.01	24.76

¹: Pine wood, source [Lekhnitskii, 1950].

²: Carbon/Epoxy T300/5208, source [Tsai and Hahn, 1980].

³: Boron/Epoxy B(4)-55054, source [Tsai and Hahn, 1980].

⁴: S-Glass/Epoxy S2-449/SP 381, source [AAVV, 2002].

⁵: Kevlar/Epoxy 149, source [Daniel and Ishai, 1994].

⁶: Ice of the Mendenhall Glacier, 270°K, source [Landolt and Börnstein, 1992].

⁷: Titanium Boride, source [Landolt and Börnstein, 1992].

⁸: Carbon/Epoxy balanced fabric, source [Gay, 2014].

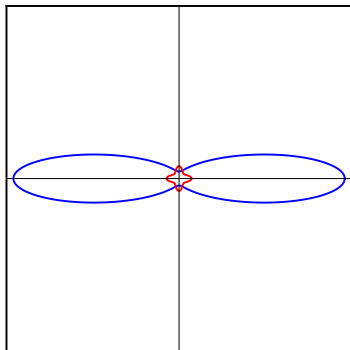
⁹: Glass/Epoxy balanced fabric 7781/5245C, source [Daniel and Ishai, 1994].

¹⁰: Braided Carbon/Epoxy BR45a, source [Falzon and Herszberg, 1998].

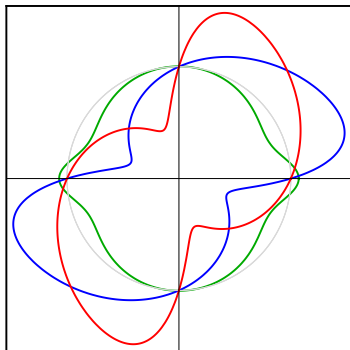
¹¹: Braided Carbon/Epoxy BR60, source [Falzon and Herszberg, 1998].

Moduli are in GPa, Q_{ij} s are in the Kelvin's notation.

Pine-wood

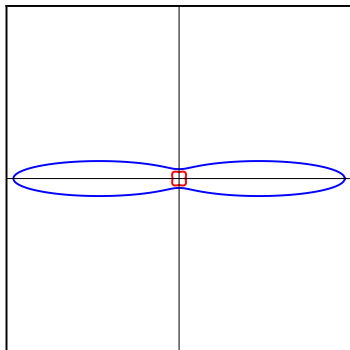


E_1, G_{12}

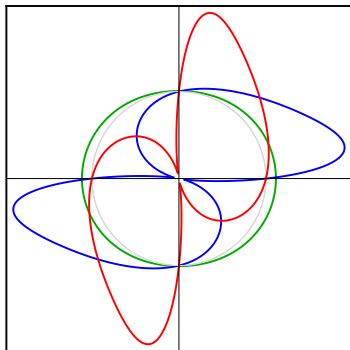


$\nu_{12}, \eta_{12,1}, \eta_{12,2}$

Carbon/Epoxy T300/5208

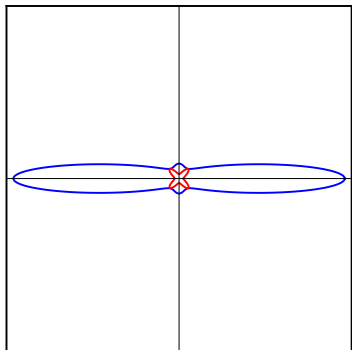


E_1, G_{12}

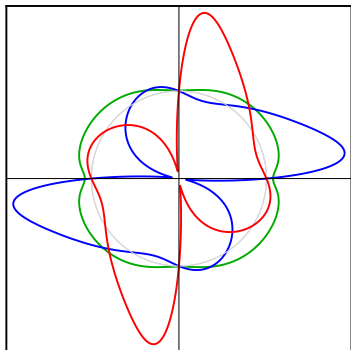


$\nu_{12}, \eta_{12,1}, \eta_{12,2}$

Boron/Epoxy B(4)-55054

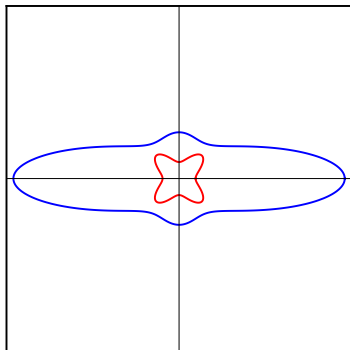


E_1, G_{12}

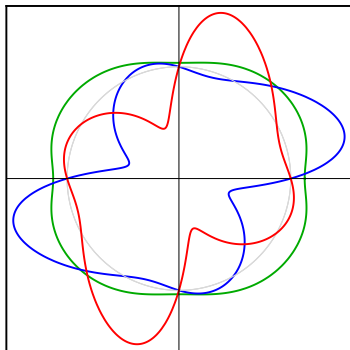


$\nu_{12}, \eta_{12,1}, \eta_{12,2}$

S-Glass/Epoxy S2-449/SP 381

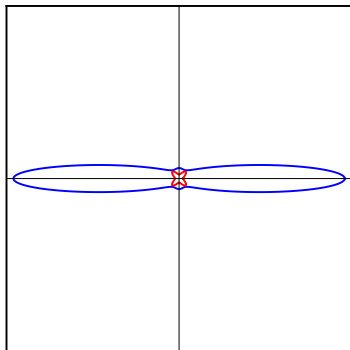


E_1 , G_{12}

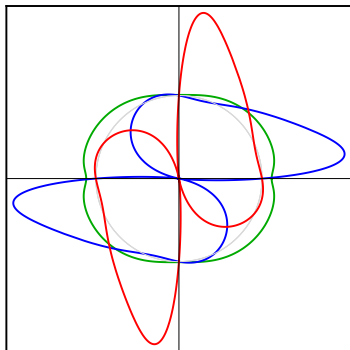


ν_{12} , $\eta_{12,1}$, $\eta_{12,2}$

Kevlar/Epoxy 149

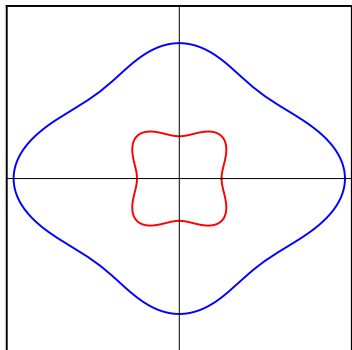


E_1, G_{12}

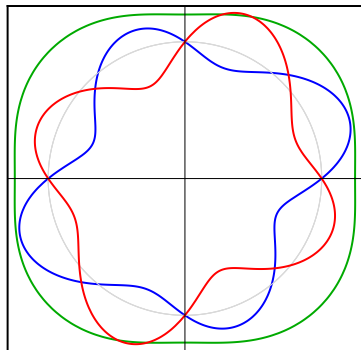


$\nu_{12}, \eta_{12,1}, \eta_{12,2}$

Ice of the Mendenhall Glacier, 270°K

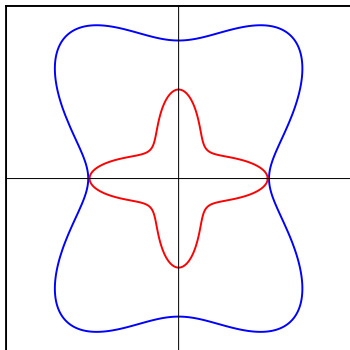


E_1, G_{12}

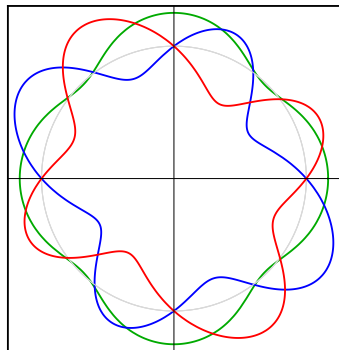


$\nu_{12}, \eta_{12,1}, \eta_{12,2}$

Titanium Boride (TiB_2)

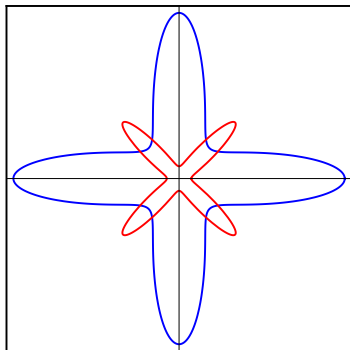


E_1, G_{12}

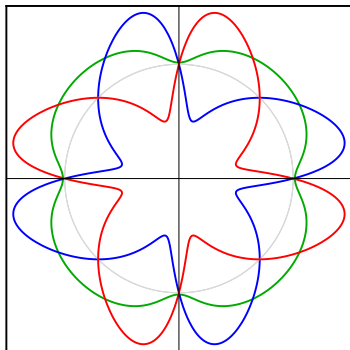


$\nu_{12}, \eta_{12,1}, \eta_{12,2}$

Carbon/Epoxy balanced fabric

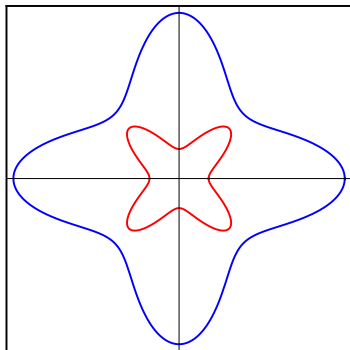


E_1, G_{12}

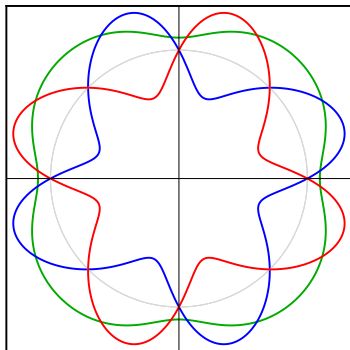


$\nu_{12}, \eta_{12,1}, \eta_{12,2}$

Glass/Epoxy 7781/5245C balanced fabric

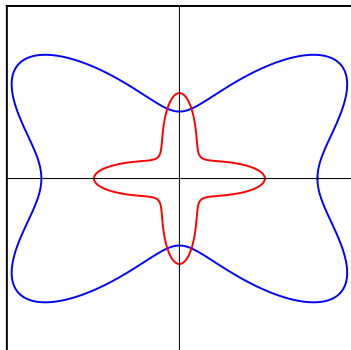


E_1, G_{12}

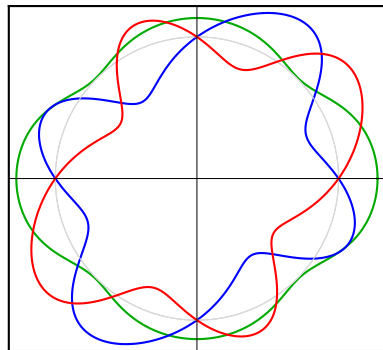


$\nu_{12}, \eta_{12,1}, \eta_{12,2}$

Braided Carbon/Epoxy BR45a

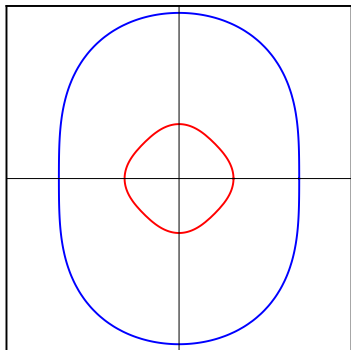


E_1 , G_{12}

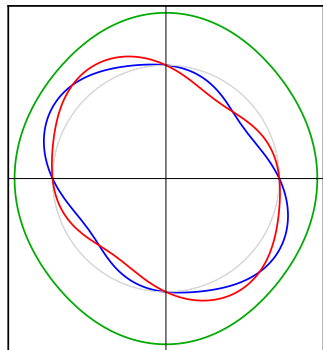


ν_{12} , $\eta_{12,1}$, $\eta_{12,2}$

Braided Carbon/Epoxy BR60



E_1, G_{12}



$\nu_{12}, \eta_{12,1}, \eta_{12,2}$

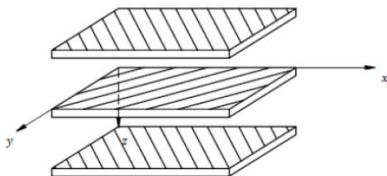
A short introduction to laminated anisotropic structures

Laminates

Laminates are plates (or shells) obtained bonding together different plies

The orientation of each layer can be chosen so as to obtain particular elastic properties

This gives a **large panel of possibilities** to designers, but also some **unexpected mechanical phenomena**



As a result, the design of laminates is a **cumbersome task**

Usually, engineers make use of some simple, semi-empirical rules to design laminated structures

The drawback is that the final design is **almost never a true optimal design**

Modern design approaches make use of **structural optimal strategies**

We consider in the following some classical results and methods for the design of the general elastic properties of laminates

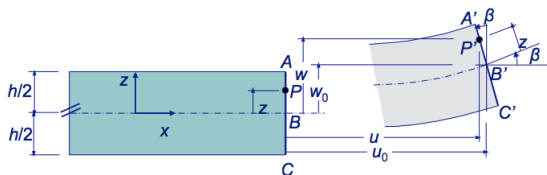
The Classical Laminated Plates Theory (CLPT)

The CLPT is based upon the classical model of **Kirchhoff**, adapted to plates composed by plies bonded together

The fundamental assumptions are

- plies perfectly bonded together → **no slip allowed**
- **linearly elastic** anisotropic plies
- **small** displacements, rotations and strains
- **small thickness** compared to a characteristic in-plane dimension
- **Kirchhoff kinematical model**: each material straight segment originally orthogonal to the midplane remains:
 - i. a **straight** segment
 - ii. **normal** to the deformed mid surface
 - iii. a segment of the **same original length**

Displacement field



For the first and third assumptions of Kirchhoff, the displacement of a point $P(x_1, x_2, x_3)$ is:

- along x_1 (along x_2 the result is similar):

$$u(x_1, x_2, x_3) = u_0(x_1, x_2) - x_3 \sin \beta \quad (41)$$

- along x_3

$$w(x_1, x_2, x_3) = w_0(x_1, x_2) + x_3(\cos \beta - 1) \quad (42)$$

For the second Kirchhoff assumption and for the hypotheses of small displacements and rotations:

$$\beta \simeq \sin \beta \simeq \tan \beta = \frac{\partial w_0}{\partial x_1}, \quad \cos \beta \simeq 1 \quad (43)$$

As a consequence

$$\begin{aligned} u(x_1, x_2, x_3) &= u_0(x_1, x_2) - x_3 \frac{\partial w_0(x_1, x_2)}{\partial x_1} \\ v(x_1, x_2, x_3) &= v_0(x_1, x_2) - x_3 \frac{\partial w_0(x_1, x_2)}{\partial x_2} \\ w(x_1, x_2) &= w_0(x_1, x_2) \end{aligned} \quad (44)$$

The kinematics of Kirchhoff is **linear with x_3**

Strain field

$$\boldsymbol{\varepsilon} = \frac{\nabla \mathbf{u} + \nabla^T \mathbf{u}}{2} \rightarrow \quad (45)$$

$$\begin{aligned}\varepsilon_{11} &= \frac{\partial u_0(x_1, x_2)}{\partial x_1} - x_3 \frac{\partial^2 w_0(x_1, x_2)}{\partial x_1^2} \\ \varepsilon_{12} &= \frac{1}{2} \left[\frac{\partial u_0(x_1, x_2)}{\partial x_2} + \frac{\partial v_0(x_1, x_2)}{\partial x_1} \right] - x_3 \frac{\partial^2 w_0(x_1, x_2)}{\partial x_1 \partial x_2} \\ \varepsilon_{22} &= \frac{\partial v_0(x_1, x_2)}{\partial x_2} - x_3 \frac{\partial^2 w_0(x_1, x_2)}{\partial x_2^2}\end{aligned} \quad (46)$$

$$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$$

The kinematics of Kirchhoff gives hence a **plane strain tensor** but **not a plane strain state** because $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(x_1, x_2, x_3)$.

Decomposition of the strain field

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^0 + x_3 \boldsymbol{\kappa} \quad \rightarrow \quad \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix} + x_3 \begin{Bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{Bmatrix} \quad (47)$$

where (Kelvin's notation)

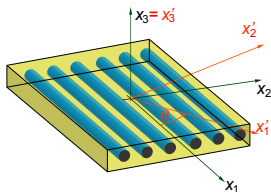
- midplane extension strain tensor

$$\boldsymbol{\varepsilon}_0 = \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0(x_1, x_2)}{\partial x_1} \\ \frac{\partial v_0(x_1, x_2)}{\partial x_2} \\ \frac{1}{\sqrt{2}} \left[\frac{\partial u_0(x_1, x_2)}{\partial x_2} + \frac{\partial v_0(x_1, x_2)}{\partial x_1} \right] \end{Bmatrix} \quad (48)$$

- curvature tensor

$$\boldsymbol{\kappa} = \begin{Bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{Bmatrix} = - \begin{Bmatrix} \frac{\partial^2 w_0(x_1, x_2)}{\partial x_1^2} \\ \frac{\partial^2 w_0(x_1, x_2)}{\partial x_2^2} \\ \sqrt{2} \frac{\partial^2 w_0(x_1, x_2)}{\partial x_1 \partial x_2} \end{Bmatrix} \quad (49)$$

The constitutive law



The basic assumption is that of layers that are **transversely isotropic** with the axis x_1 , see the figure, as symmetry axis:

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{22} - C_{23}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \quad (50)$$

It is easily seen that in a frame rotated through an angle θ it is

$$\begin{aligned}C'_{13} &= C_{12} \cos^2 \theta + C_{23} \sin^2 \theta, \\C'_{23} &= C_{12} \sin^2 \theta + C_{23} \cos^2 \theta, \\C'_{63} &= \sqrt{2}(C_{23} - C_{12}) \sin \theta \cos \theta, \\C'_{14} &= C'_{24} = C'_{64} = C'_{15} = C'_{25} = C'_{65} = 0.\end{aligned}\tag{51}$$

Then

$$\begin{aligned}\sigma'_4 &= \sigma'_5 = 0, \\ \sigma'_3 &= (C_{12} \cos^2 \theta + C_{23} \sin^2 \theta)\varepsilon'_1 + (C_{12} \sin^2 \theta + C_{23} \cos^2 \theta)\varepsilon'_2 + \\ &\quad \sqrt{2}(C_{23} - C_{12}) \sin \theta \cos \theta \varepsilon'_6.\end{aligned}\tag{52}$$

Generally speaking, $\sigma'_3 \neq 0 \rightarrow$ the stress state is **not plane**

The stress field

What is commonly admitted, on a heuristic base, is that $\sigma_3 = 0$.

This in the end corresponds to admit that the stress field is plane too

Nevertheless, one should notice that unlike in the case of a true plane stress field, both the stress and strain fields are not plane functions:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^P(x_1, x_2, x_3), \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^P(x_1, x_2, x_3) \quad (53)$$

So, this case does not corresponds to any of the cases seen before: plane strain or stress, generalized plane stress or strain, plane deformation.

In particular, contrarily to what is commonly said, it is not a generalized plane stress state.

Finally, the constitutive law is that of a plane stress state, with the stiffness matrix $[C]$ that is replaced by the reduced stiffness matrix $[Q]$

In the material (orthotropy) frame,

$$\{\sigma\} = [Q]\{\varepsilon\} \rightarrow \{\sigma\} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} \quad (54)$$

where

$$Q_{ij} = C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}}, \quad i, j = 1, 2, 6 \rightarrow \quad (55)$$

$$[Q] = \begin{bmatrix} C_{11} - \frac{C_{13}^2}{C_{33}} & C_{12} - \frac{C_{13}C_{23}}{C_{33}} & 0 \\ & C_{22} - \frac{C_{23}^2}{C_{33}} & 0 \\ \text{sym} & & C_{66} \end{bmatrix}. \quad (56)$$

To remark that in a general frame $[Q]$ is **full**

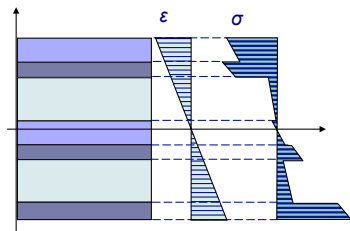
The result of the constitutive law for the special case considered and of the Kirchhoff kinematics is that $\sigma_4 = \sigma_5 = 0$ in any frame

A consequence of this fact is that equilibrium is possible only for extension and pure bending, but **not for shearing actions** (i.e. loads normal to the midplane cannot be equilibrated)

This inconsistency, typical of the Kirchhoff theory, is solved in the case of single layer plates using the **equilibrium equations**

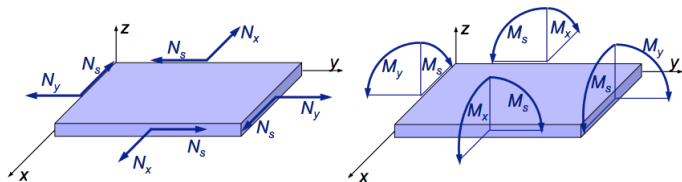
This is **not** possible in general for laminates

The use of higher order theories only can solve this problem for laminates



To be remarked that unlike the strain field ϵ , the stress field σ is **not continuous at the interfaces**, due to the change of constitutive law from a ply to another one, see the figure.

The internal actions



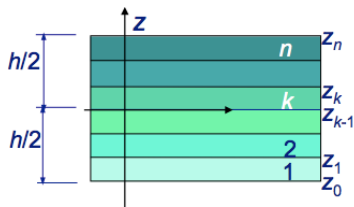
We define the tensors of **in-plane actions** **N** and of **bending moments** **M**

$$\mathbf{N} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \boldsymbol{\sigma} dx_3, \quad \mathbf{M} = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3 \boldsymbol{\sigma} dx_3 \rightarrow$$
$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} = \begin{Bmatrix} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_1 dx_3 \\ \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_2 dx_3 \\ \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_6 dx_3 \end{Bmatrix}, \quad \begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix} = \begin{Bmatrix} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_1 x_3 dx_3 \\ \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_2 x_3 dx_3 \\ \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_6 x_3 dx_3 \end{Bmatrix} \quad (57)$$

Injecting the constitutive law and considering that, being this different layer by layer, the integrals must be split into the sum of n terms, we get (δ_k is the orientation of the k -th ply)

$$\mathbf{N} = \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \boldsymbol{\sigma}_k dx_3 = \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \mathbb{Q}_k(\delta_k)(\varepsilon_0 + x_3 \boldsymbol{\kappa}) dx_3 \quad (58)$$

$$\mathbf{M} = \sum_{k=1}^n \int_{z_{k-1}}^{z_k} x_3 \boldsymbol{\sigma}_k dx_3 = \sum_{k=1}^n \int_{z_{k-1}}^{z_k} x_3 \mathbb{Q}_k(\delta_k)(\varepsilon_0 + x_3 \boldsymbol{\kappa}) dx_3 \quad (59)$$



The laminates constitutive law

We now introduce the tensors

- \mathbb{A} , tensor of the **extension behavior**

$$\mathbb{A} = \frac{1}{h} \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \mathbb{Q}_k(\delta_k) dx_3 = \frac{1}{h} \sum_{k=1}^n (z_k - z_{k-1}) \mathbb{Q}_k(\delta_k) \quad (60)$$

- \mathbb{B} , tensor of the **coupling behavior**

$$\mathbb{B} = \frac{2}{h^2} \sum_{k=1}^n \int_{z_{k-1}}^{z_k} x_3 \mathbb{Q}_k(\delta_k) dx_3 = \frac{1}{h^2} \sum_{k=1}^n (z_k^2 - z_{k-1}^2) \mathbb{Q}_k(\delta_k) \quad (61)$$

- \mathbb{D} , tensor of the **bending behavior**

$$\mathbb{D} = \frac{12}{h^3} \sum_{k=1}^n \int_{z_{k-1}}^{z_k} x_3^2 \mathbb{Q}_k(\delta_k) dx_3 = \frac{4}{h^3} \sum_{k=1}^n (z_k^3 - z_{k-1}^3) \mathbb{Q}_k(\delta_k) \quad (62)$$

Then we obtain the following constitutive relations

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix} = \begin{bmatrix} h\mathbb{A} & \frac{h^2}{2}\mathbb{B} \\ \frac{h^2}{2}\mathbb{B} & \frac{h^3}{12}\mathbb{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{Bmatrix} \quad (63)$$

This is the **fundamental law of laminates**, linking the internal actions to the laminate's deformation.

By construction,

$$\mathbb{A} = \mathbb{A}^T, \quad \mathbb{B} = \mathbb{B}^T, \quad \mathbb{D} = \mathbb{D}^T \quad (64)$$

In this way, the mechanics of a multi-layer plate is reduced to that of an **equivalent single layer** plate

Nevertheless, there are some special **differences** with respect to a single layer plate...

Heterogeneity of the elastic behavior

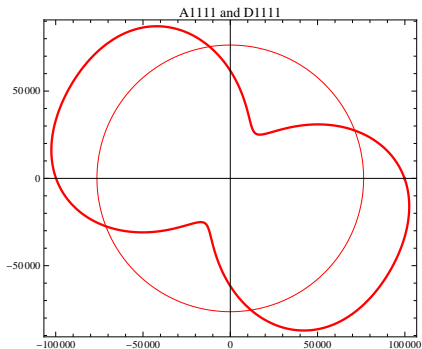
A first difference is the **heterogeneity of the elastic behavior**:
generally speaking,

$$\mathbb{A} \neq \mathbb{D} \quad (65)$$

The laminate has a **different** elastic response, at any direction, in extension and in bending, like if it were composed by two different materials

In the figure, an example with the components \mathbb{A}_{11} and \mathbb{D}_{11} for a same laminate

The first one is **isotropic**, the second one **orthotropic**



Bending-extension coupling

A second difference is the existence of a **coupling** between extension and bending

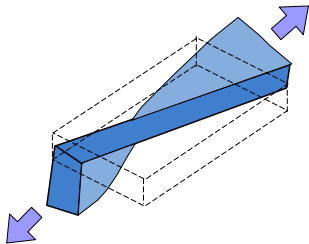
In fact, generally speaking,

$$\mathbb{B} \neq 0 \quad (66)$$

So, an extension produces also a curvature of the plate while a bending moment stretches the midplane

A laminate with $\mathbb{B} = 0$ is said to be **uncoupled**; in this case, it is simply

$$\mathbf{N} = h\mathbf{A}\boldsymbol{\varepsilon}^0, \quad \mathbf{M} = \frac{h^3}{12}\mathbb{D}\boldsymbol{\kappa} \quad (67)$$



Quasi-homogeneous laminates

The concept of quasi-homogeneous laminate has been introduced by G. Verchery in 1988.

We define the **homogeneity tensor**

$$\mathbb{C} = \mathbb{A} - \mathbb{D} \quad (68)$$

Then, a laminate is said to be **quasi-homogeneous** \iff

$$\mathbb{B} = \mathbb{C} = 0 \quad (69)$$

In such a case, the elastic behavior is the same in bending and in extension and there is no coupling

The laminate has hence a behavior like that of a homogeneous, i.e. single layer, plate, that's why the name quasi-homogeneous

Inverting the constitutive law of laminates

$$\mathbf{M} = \frac{h^2}{2} \mathbb{B} \boldsymbol{\varepsilon}^0 + \frac{h^3}{12} \mathbb{D} \boldsymbol{\kappa} \rightarrow \boldsymbol{\kappa} = \frac{12}{h^3} \mathbb{D}^{-1} (\mathbf{M} - \frac{h^2}{2} \mathbb{B} \boldsymbol{\varepsilon}^0) \quad (70)$$

$$\mathbf{N} = h \mathbb{A} \boldsymbol{\varepsilon}^0 + \frac{h^2}{2} \mathbb{B} \boldsymbol{\kappa} \rightarrow \mathbf{N} = h \mathbb{A} \boldsymbol{\varepsilon}^0 + \frac{h^2}{2} \mathbb{B} \frac{12}{h^3} \mathbb{D}^{-1} (\mathbf{M} - \frac{h^2}{2} \mathbb{B} \boldsymbol{\varepsilon}^0) \quad (71)$$

Hence, resolving the last equation with respect to $\boldsymbol{\varepsilon}^0$,

$$\boldsymbol{\varepsilon}^0 = \frac{1}{h} \mathcal{A} \mathbf{N} + \frac{2}{h^2} \mathcal{B}_1 \mathbf{M} \quad (72)$$

In a similar way we get also

$$\boldsymbol{\kappa} = \frac{2}{h^2} \mathcal{B}_2 \mathbf{N} + \frac{12}{h^3} \mathcal{D} \mathbf{M} \quad (73)$$

with

$$\begin{aligned} \mathcal{A} &= (\mathbb{A} - 3\mathbb{B}\mathbb{D}^{-1}\mathbb{B})^{-1}, & \mathcal{B}_1 &= -3\mathcal{A}\mathbb{B}\mathbb{D}^{-1}, \\ \mathcal{D} &= (\mathbb{D} - 3\mathbb{B}\mathcal{A}^{-1}\mathbb{B})^{-1}, & \mathcal{B}_2 &= -3\mathcal{D}\mathbb{B}\mathcal{A}^{-1}. \end{aligned} \quad (74)$$

We find now another **strange behavior** of laminates

In fact, it is easy to check that

$$\mathcal{A}^\top = \left((\mathbb{A} - 3\mathbb{B}\mathbb{D}^{-1}\mathbb{B})^\top \right)^{-1} = (\mathbb{A} - 3\mathbb{B}\mathbb{D}^{-1}\mathbb{B})^{-1} = \mathcal{A} \quad (75)$$

and similarly

$$\mathcal{D}^\top = \mathcal{D} \quad (76)$$

but

$$\begin{aligned} \mathcal{B}_1 &\neq \mathcal{B}_2, \\ \mathcal{B}_1^\top &= (-3\mathcal{A}\mathbb{B}\mathbb{D}^{-1})^\top = -3\mathbb{D}^{-1}\mathbb{B}\mathcal{A} \neq \mathcal{B}_1, \\ \mathcal{B}_2^\top &= (-3\mathcal{D}\mathbb{B}\mathbb{A}^{-1})^\top = -3\mathbb{A}^{-1}\mathbb{B}\mathcal{D} \neq \mathcal{B}_2, \end{aligned} \quad (77)$$

Hence, \mathcal{B}_1 and \mathcal{B}_2 are not only different, but **asymmetric too** $\rightarrow \mathbb{B}$ is **not a classical elastic tensor**, because its compliance corresponding are not unique and without the major symmetries.

In addition, \mathbb{B} is **not definite**, and no bounds can be given to its components

Nevertheless, \mathcal{B}_1 and \mathcal{B}_2 are **not independent**:

$$\begin{aligned}(\mathcal{B}_1^\top)^{-1} &= -\frac{1}{3}\mathcal{A}^{-1}\mathbb{B}^{-1}\mathbb{D} = -\frac{1}{3}\mathbb{A}\mathbb{B}^{-1}\mathbb{D} + \mathbb{B}, \\ \mathcal{B}_2^{-1} &= -\frac{1}{3}\mathbb{A}\mathbb{B}^{-1}\mathbb{D}^{-1} = -\frac{1}{3}\mathbb{A}\mathbb{B}^{-1}\mathbb{D} + \mathbb{B} = (\mathcal{B}_1^\top)^{-1}.\end{aligned}\tag{78}$$

Hence, for the uniqueness of the inverse,

$$\mathcal{B}_2^\top = \mathcal{B}_1 := \mathcal{B},\tag{79}$$

so that if we write the inverse law in the form

$$\left\{ \begin{array}{c} \varepsilon^0 \\ \kappa \end{array} \right\} = \left[\begin{array}{c|c} \frac{1}{h}\mathcal{A} & \frac{2}{h^2}\mathcal{B} \\ \frac{2}{h^2}\mathcal{B}^\top & \frac{12}{h^3}\mathcal{D} \end{array} \right] \left\{ \begin{array}{c} \mathbf{N} \\ \mathbf{M} \end{array} \right\}\tag{80}$$

then the global compliance matrix preserves a **symmetry** with respect to the main diagonal

As said, while $\mathbb{B} = \mathbb{B}^T$, this is not true for compliance: $\mathcal{B} \neq \mathcal{B}^T$.

The question is: which is the condition for having also $\mathcal{B} \neq \mathcal{B}^T$?

It can be shown that, for laminates with identical plies, if $\mathbb{B} \neq 0$ but $\mathbb{A}, \mathbb{B}, \mathbb{D}$ are orthotropic, then (J Opt Th Appl, 2013)

$$\frac{(-1)^{K^A} R_0^A - (-1)^{K^D} R_0^D}{R_1^A - R_1^D} = \frac{(-1)^{K^B} R_0^B}{R_1^B} \Rightarrow \mathcal{B} = \mathcal{B}^T$$

This elegant formula shows the role of all the polar invariants; the different tensors appear in the relation that has a strong algebraic symmetry.

The true reason of this is not clear yet but shows, anyway, the algebraic effectiveness of the polar formalism.

We remark that for uncoupled laminates, and only in this case,

$$\mathbb{B} = 0 \iff \mathcal{B} = 0 \Rightarrow \mathcal{A} = \mathbb{A}^{-1}, \quad \mathcal{D} = \mathbb{D}^{-1} \quad (81)$$

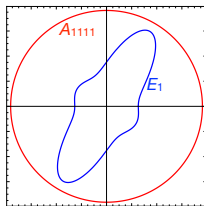
Only in this case, the symmetries of \mathbb{A} or of \mathbb{D} are the **same in stiffness and in compliance**

In all the other cases, i.e. when a laminate is coupled, the stiffness and compliance tensors of the extension and bending behaviors have, generally speaking, **different symmetries**

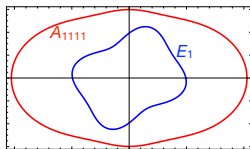
This is another strange fact of laminates with respect to single layer plates and corroborate, once more, the idea that the symmetry of the elastic behavior is first of all a **matter of algebraic properties of an elastic tensor**, more than a material symmetry of the plate, that in the case of coupled laminates cannot be clearly defined

Just two examples concerning laminates composed of identical plies, see below.

- If $\mathbb{B} \neq 0$, the symmetries of \mathbb{A} and \mathbb{D} are lost for \mathcal{A} and \mathcal{D}



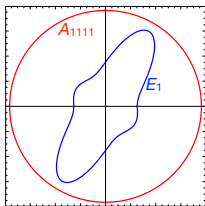
$[0/60/120]$



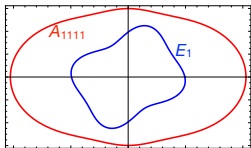
$[0/30/90/150]$

Just two examples concerning laminates composed of identical plies, see below.

- If $\mathbb{B} \neq 0$, the symmetries of \mathbb{A} and \mathbb{D} are lost for \mathcal{A} and \mathcal{D}

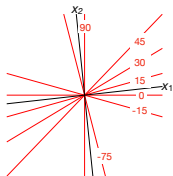
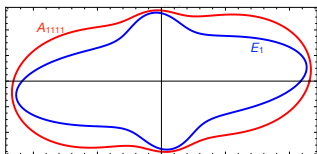


$[0/60/120]$



$[0/30/90/150]$

- An elastic symmetry can exist also without any material symmetry. An example with \mathbb{A} orthotropic, $\mathbb{B} = 0$, \mathbb{D} anisotropic:
 $[0/30/-15/15/90/-75/0/45/-75/0/-15/15]$



Laminates with identical plies

If all the plies are identical, i.e. **same material and thickness**, then

$$z_k = \frac{2k - n}{2n} h \quad (82)$$

and the equations giving \mathbb{A} , \mathbb{B} , \mathbb{C} and \mathbb{D} become

$$\begin{aligned} \mathbb{A} &= \sum_{k=1}^n a_k \mathbb{Q}(\delta_k), & \mathbb{B} &= \sum_{k=1}^n b_k \mathbb{Q}(\delta_k), \\ \mathbb{C} &= \sum_{k=1}^n c_k \mathbb{Q}(\delta_k), & \mathbb{D} &= \sum_{k=1}^n d_k \mathbb{Q}(\delta_k), \end{aligned} \quad (83)$$

where

$$\begin{aligned} a_k &= \frac{1}{n}, & b_k &= \frac{1}{n^2}(2k - n - 1), & c_k &= a_k - d_k, \\ d_k &= \frac{1}{n^3} [12k(k - n - 1) + 4 + 3n(n + 2)], \end{aligned} \quad (84)$$

Because $a_k = \frac{1}{n} \forall k$, then the **stacking sequence does not affect the extension behavior**, which is determined only by the orientations

This is **not** the case of coupling and bending, where the stacking sequence influences the final behavior, with the orientations

This is why it is much **easier to design with respect to extension** than to bending

Laminates by the polar formalism

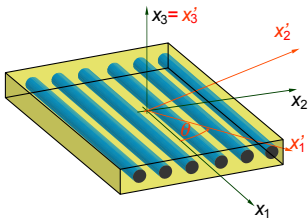
The polar formalism is very helpful in the design problems of laminates, for different reasons:

- the polar invariants help in representing effectively symmetries and in stating them
- some of the polar invariants are preserved by the laminate
- the separation between the isotropic and anisotropic phases is of the primary importance: it allows for better understanding the design process and to eliminate redundant design variables

Basically, the mechanics of laminates does not change, of course, nor the general results for laminate tensors

It is just the representation of these tensors which is different, but **not** the way they are constructed: the homogenization laws of the laminate tensors are independent from the elasticity algebraic representation → **they apply to the polar formalism as well**

In the end, we get (each ply is rotated of $-\delta_k$ with respect to the global (red) frame, see the figure)



$$\mathbb{A} \rightarrow \left\{ \begin{array}{l} T_0^A = \frac{1}{h} \sum_{k=1}^n T_{0k}(z_k - z_{k-1}) \\ T_1^A = \frac{1}{h} \sum_{k=1}^n T_{1k}(z_k - z_{k-1}) \\ R_0^A e^{4i\Phi_0^A} = \frac{1}{h} \sum_{k=1}^n R_{0k} e^{4i(\Phi_{0k} + \delta_k)} (z_k - z_{k-1}) \\ R_1^A e^{2i\Phi_1^A} = \frac{1}{h} \sum_{k=1}^n R_{1k} e^{2i(\Phi_{1k} + \delta_k)} (z_k - z_{k-1}) \end{array} \right. \quad (85)$$

$$\mathbb{B} \rightarrow \left\{ \begin{array}{l} T_0^B = \frac{1}{h^2} \sum_{k=1}^n T_{0k}(z_k^2 - z_{k-1}^2) \\ T_1^B = \frac{1}{h^2} \sum_{k=1}^n T_{1k}(z_k^2 - z_{k-1}^2) \\ R_0^B e^{4i\Phi_0^B} = \frac{1}{h^2} \sum_{k=1}^n R_{0k} e^{4i(\Phi_{0k} + \delta_k)}(z_k^2 - z_{k-1}^2) \\ R_1^B e^{2i\Phi_1^B} = \frac{1}{h^2} \sum_{k=1}^n R_{1k} e^{2i(\Phi_{1k} + \delta_k)}(z_k^2 - z_{k-1}^2) \end{array} \right. \quad (86)$$

$$\mathbb{D} \rightarrow \left\{ \begin{array}{l} T_0^D = \frac{4}{h^3} \sum_{k=1}^n T_{0k} (z_k^3 - z_{k-1}^3) \\ T_1^D = \frac{4}{h^3} \sum_{k=1}^n T_{1k} (z_k^3 - z_{k-1}^3) \\ R_0^D e^{4i\Phi_0^D} = \frac{4}{h^3} \sum_{k=1}^n R_{0k} e^{4i(\Phi_{0k} + \delta_k)} (z_k^3 - z_{k-1}^3) \\ R_1^D e^{2i\Phi_1^D} = \frac{4}{h^3} \sum_{k=1}^n R_{1k} e^{2i(\Phi_{1k} + \delta_k)} (z_k^3 - z_{k-1}^3) \end{array} \right. \quad (87)$$

Some remarks:

- the isotropic and anisotropic parts of all the tensors remain **separated** in the homogenization of the polar parameters, for all the tensors
- it is immediately apparent that **special orthotropies are preserved**:

$$\begin{aligned} R_{0k} = 0 \quad \forall k &\Rightarrow R_0^A = R_0^B = R_0^C = R_0^D = 0 \\ R_{1k} = 0 \quad \forall k &\Rightarrow R_1^A = R_1^B = R_1^C = R_1^D = 0 \end{aligned} \quad (88)$$

More results are obtained for **laminates of identical plies** ...