International Doctorate in Civil and Environmental Engineering

Anisotropic Structures - Theory and Design

Strutture anisotrope: teoria e progetto

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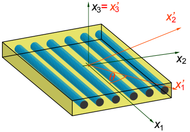

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Topics of the third lesson

- Plane problems

The planar case

In a great number of situations the problem can be reduced from a 3D to a planar one, because of its geometry and loading conditions.

This reduction can considerably simplify the problem and also opens the way to the use of special mathematical techniques, like for instance complex variables.

Actually, different cases can be considered; to this purpose, it is worth to make a distinction between

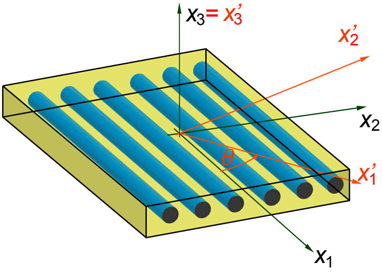
- *plane tensor*: it is a tensor whose components orthogonal to a given plane, say the plane $x_3 = 0$, are all null (i.e. $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$, $\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$)
- *plane field*: it is a tensor function whose components are scalar functions independent of x_3: $\sigma_{ij} = \sigma_{ij}(x_1, x_2), \varepsilon_{ij} = \varepsilon_{ij}(x_1, x_2),\ \forall i, j = 1, 2, 3.$

A plane field is, hence, not necessarily a plane tensor, and cases are possible, depending on the assumptions, where one of the tensors is not plane nor a plane field, while the others are plane tensors and/or plane fields.

The possible combinations are different, and the literature is not always completely clear about this topic.

In the following, an exposition as complete as possible is given, considering the different approaches and the possible definitions existing in the literature:

- plane strain
- plane stress
- generalized plane stress
- the Lekhnitskii's theory
- the Stroh's theory



The figure shows the general sketch

- the structure belongs to the plane $x_3 = 0$
- basis $\mathcal{B} = \{x_1, x_2, x_3\}$ is the *material basis*, where the properties of the material are known (typically, the orthotropic basis)
- basis, $\mathcal{B}' = \{x_1', x_2', x_3'\}$ is a generic basis, rotated counterclockwise through an angle θ about the axis $x_3 = x_3'$

Rotation of the axes in 2D

The change from basis $\mathcal{B} = \{x_1, x_2\}$ to $\mathcal{B}' = \{x_1', x_2'\}$, sketched in the Figure, is represented by the orthogonal tensor

$$\mathbf{U} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad c = \cos\theta, \; s = \sin\theta, \tag{1}$$

which gives the rotation matrix for 2D problems

$$[R] = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & \sqrt{2}cs \\ s^2 & c^2 & 0 & 0 & 0 & -\sqrt{2}cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -\sqrt{2}cs & \sqrt{2}cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix}. \tag{2}$$

Extracting the plane components of $\{\varepsilon\}$, i.e. considering in matrix (2) the relevant components, then

$$\{\varepsilon\}' = [R]\{\varepsilon\} \rightarrow \left\{ \begin{array}{c} \varepsilon_1' \\ \varepsilon_2' \\ \varepsilon_6' \end{array} \right\} = \left[\begin{array}{ccc} c^2 & s^2 & \sqrt{2}cs \\ s^2 & c^2 & -\sqrt{2}cs \\ -\sqrt{2}cs & \sqrt{2}cs & c^2 - s^2 \end{array} \right] \left\{ \begin{array}{c} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{array} \right\} \tag{3}$$

and

$$[S'] = [R][S][R]^\top \rightarrow$$

$$\left\{ \begin{array}{c} S_{11}' \\ S_{16}' \\ S_{12}' \\ S_{66}' \\ S_{26}' \\ S_{22}' \end{array} \right\} = \left[\begin{array}{cccccc} c^4 & 2\sqrt{2}c^3 s & 2c^2s^2 & 2c^2s^2 & 2\sqrt{2}cs^3 & s^4 \\ -\sqrt{2}c^3 s & c^4 - 3c^2s^2 & \sqrt{2}cs(c^2 - s^2) & \sqrt{2}cs(c^2 - s^2) & 3c^2s^2 - s^4 & \sqrt{2}cs^3 \\ c^2s^2 & \sqrt{2}cs(s^2 - c^2) & c^4 + s^4 & -2c^2s^2 & \sqrt{2}cs(c^2 - s^2) & c^2s^2 \\ 2c^2s^2 & 2\sqrt{2}cs(s^2 - c^2) & -4c^2s^2 & (c^2 - s^2)^2 & 2\sqrt{2}cs(c^2 - s^2) & 2c^2s^2 \\ -\sqrt{2}cs^3 & 3c^2s^2 - s^4 & \sqrt{2}cs(s^2 - c^2) & \sqrt{2}cs(s^2 - c^2) & c^4 - 3c^2s^2 & \sqrt{2}c^3 s \\ s^4 & -2\sqrt{2}cs^3 & 2c^2s^2 & 2c^2s^2 & -2\sqrt{2}c^3 s & c^4 \end{array} \right] \left\{ \begin{array}{c} S_{11} \\ S_{16} \\ S_{12} \\ S_{66} \\ S_{26} \\ S_{22} \end{array} \right\} \tag{4}$$

These are the transformation matrices for ε and $[S]$ in 2D. Similar results are valid for $\{\sigma\}$ and $[C]$ (Kelvin's notation)

The Tsai and Pagano parameters

Tsai and Pagano (1968) proposed a transformation of eq. (4), obtained exclusively using standard trigonometric identities:

$$
\left\{\begin{array}{c} Q'_{11} \\ Q'_{12} \\ Q'_{16} \\ Q'_{22} \\ Q'_{26} \\ Q'_{66} \end{array}\right\} = \left[\begin{array}{ccccccc} 1 & \cos 2\theta & \cos 4\theta & 0 & 0 & 2\sin 2\theta & \sin 4\theta \\ 0 & 0 & -\cos 4\theta & 1 & 0 & 0 & -\sin 4\theta \\ 0 & \frac{\sqrt{2}}{2}\sin 2\theta & \sqrt{2}\sin 4\theta & 0 & 0 & \sqrt{2}\cos 2\theta & \sqrt{2}\cos 4\theta \\ 1 & -\cos 2\theta & \cos 4\theta & 0 & 0 & -2\sin 2\theta & \sin 4\theta \\ 0 & \frac{\sqrt{2}}{2}\sin 2\theta & -\sqrt{2}\sin 4\theta & 0 & 0 & \sqrt{2}\cos 2\theta & -\sqrt{2}\cos 4\theta \\ 0 & 0 & -2\cos 4\theta & 0 & 2 & 0 & -2\sin 4\theta \end{array}\right] \left\{\begin{array}{c} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \end{array}\right\} \tag{5}
$$

$[Q]$: reduced stiffness matrix of a plane stress state (see below).

The original transformation, written for the Voigt's notation, is slightly different and valid only for $[Q]$, while eq. (5) can be applied to $[S]$ too.

U_i: Tsai and Pagano parameters, linear combinations of the components of the matrix in the original frame:

$$
\begin{aligned}
U_1 &= \frac{1}{8}(3Q_{11} + 2Q_{12} + 3Q_{22} + 2Q_{66}),\\
U_2 &= \frac{1}{2}(Q_{11} - Q_{22}),\\
U_3 &= \frac{1}{8}(Q_{11} - 2Q_{12} + Q_{22} - 2Q_{66}),\\
U_4 &= \frac{1}{8}(Q_{11} + 6Q_{12} + Q_{22} - 2Q_{66}),\\
U_5 &= \frac{1}{8}(Q_{11} - 2Q_{12} + Q_{22} + 2Q_{66}),\\
U_6 &= \frac{1}{2\sqrt{2}}(Q_{16} + Q_{26}),\\
U_7 &= \frac{1}{2\sqrt{2}}(Q_{16} - Q_{26}).
\end{aligned}
\tag{6}
$$

In the literature, the U_is are often called invariants, like in the same title of the original publication, but this is not correct: U_2, U_3, U_6 and U_7 are frame dependent quantities.

To remark that Tsai and Pagano make use of 7 quantities to express 6 other functions. As a consequence, the U_is are not all independent, e.g.

$$U_5 = \frac{U_1 - U_4}{2}. \tag{7}$$

Th U_is have not a direct and clear physical meaning, nor they are immediately linked to the anisotropic properties or to the elastic symmetries.

Their use is exclusively limited to the design of laminates.

Recalling some classical results and tools

It is worth now to recall some classical topics in plane elasticity, to be used in the following.

Airy's stress function (1862)

Airy noticed that in 2D problems the equilibrium equations of a body subjected to only surface tractions (i.e. with a null body vector) indicate that the σ_{ij} can be regarded as the second-order partial derivatives of a single scalar function, the Airy's stress function

The knowledge of the Airy's stress function gives the $\sigma_{\alpha\beta}$ that automatically satisfy the equilibrium equations.

We give here the most general approach to the Airy's stress function, valid regardless of the type of material and including also the presence of a body vector (cf. Milne-Thomson).

Consider a plane system, for which we assume

$$\sigma_{ij} = \sigma_{ij}(x_1, x_2), \quad \sigma_{23} = \sigma_{31} = 0, \tag{8}$$

which implies that the equilibrium equations reduce to

$$\sigma_{\alpha\beta,\beta} = b_\alpha, \quad \alpha, \beta = 1, 2. \tag{9}$$

For such a plane problem, we introduce the complex variable

$$z = x_1 + ix_2 \;\; \rightarrow \;\; \bar{z} = x_1 - ix_2 \tag{10}$$

and conversely

$$x_1 = \frac{1}{2}(z + \bar{z}), \quad x_2 = -\frac{1}{2}i(z - \bar{z}). \tag{11}$$

For the differential operators we have then the following equivalences

$$\begin{cases} \dfrac{\partial}{\partial x_1} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \overline{z}}, \\ \dfrac{\partial}{\partial x_2} = i\dfrac{\partial}{\partial z} - i\dfrac{\partial}{\partial \overline{z}}, \end{cases} \qquad \begin{cases} 2\dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial x_1} - i\dfrac{\partial}{\partial x_2}, \\ 2\dfrac{\partial}{\partial \overline{z}} = \dfrac{\partial}{\partial x_1} + i\dfrac{\partial}{\partial x_2}. \end{cases} \tag{12}$$

If $(12)_1$ is injected into (9) we get

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial z} + \frac{\partial \sigma_{11}}{\partial \overline{z}} + i\left(\frac{\partial \sigma_{12}}{\partial z} - \frac{\partial \sigma_{12}}{\partial \overline{z}}\right) &= b_1, \\ \frac{\partial \sigma_{21}}{\partial z} + \frac{\partial \sigma_{21}}{\partial \overline{z}} + i\left(\frac{\partial \sigma_{22}}{\partial z} - \frac{\partial \sigma_{22}}{\partial \overline{z}}\right) &= b_2; \end{aligned} \tag{13}$$

multiplying the second equation by $-i$ and adding the result to the first equation gives

$$\frac{\partial \Theta}{\partial z} - \frac{\partial \Phi}{\partial \overline{z}} = b_1 - ib_2, \tag{14}$$

$$\Theta = \sigma_{11} + \sigma_{22}, \quad \Phi = \sigma_{22} - \sigma_{11} + 2i\sigma_{12}, \tag{15}$$

are the Kolosov's fundamental stress combinations (1909).

Be Θ_0, Φ_0 a particular solution of (14) corresponding to the action of the body vector, i.e. such that

$$\frac{\partial \Theta_0}{\partial z} - \frac{\partial \Phi_0}{\partial \overline{z}} = b_1 - ib_2; \tag{16}$$

then, the general solution of (14) is

$$\Theta = \Theta_0 + 4\frac{\partial^2 \chi}{\partial z \partial \overline{z}}, \quad \Phi = \Phi_0 + 4\frac{\partial^2 \chi}{\partial z^2}. \tag{17}$$

The arbitrary real valued function

$$\chi = \chi(x_1, x_2) = \chi(z, \overline{z}) \tag{18}$$

is the Airy's stress function.

The solution of the stress problem is hence reduced to the knowledge of the Airy's function: from eqs. (15) and (17) we get

$$
\begin{aligned}
\sigma_{11} &= \frac{1}{2}\Theta - \frac{1}{4}(\Phi + \overline{\Phi}) = \sigma_{11}^0 + \frac{\partial^2 \chi}{\partial x_2^2}, \\
\sigma_{22} &= \frac{1}{2}\Theta + \frac{1}{4}(\Phi + \overline{\Phi}) = \sigma_{22}^0 + \frac{\partial^2 \chi}{\partial x_1^2}, \\
\sigma_{12} &= -\frac{1}{4}i(\Phi - \overline{\Phi}) = \sigma_{12}^0 - \frac{\partial^2 \chi}{\partial x_1 \partial x_2},
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
\sigma_{11}^0 &= \frac{1}{2}\Theta_0 - \frac{1}{4}(\Phi_0 + \overline{\Phi}_0), \\
\sigma_{22}^0 &= \frac{1}{2}\Theta_0 + \frac{1}{4}(\Phi_0 + \overline{\Phi}_0), \\
\sigma_{12}^0 &= -\frac{1}{4}i(\Phi_0 - \overline{\Phi}_0),
\end{aligned}
\tag{20}
$$

are a particular solution of the equilibrium equations (9) accounting for the body vector.

In case of body forces depending upon a potential U, $\mathbf{f} = \nabla U$, then eq. (19) becomes

$$\begin{aligned}
\sigma_{11} &= \frac{\partial^2 \chi}{\partial x_2^2} - U, \\
\sigma_{22} &= \frac{\partial^2 \chi}{\partial x_1^2} - U, \\
\sigma_{12} &= -\frac{\partial^2 \chi}{\partial x_1 \partial x_2}.
\end{aligned} \tag{21}$$

When the body is acted upon uniquely by surface tractions, eq. (19) becomes simply

$$\begin{aligned}
\sigma_{11} &= \frac{\partial^2 \chi}{\partial x_2^2}, \\
\sigma_{22} &= \frac{\partial^2 \chi}{\partial x_1^2}, \\
\sigma_{12} &= -\frac{\partial^2 \chi}{\partial x_1 \partial x_2}.
\end{aligned} \tag{22}$$

It is possible to introduce the The Airy's function without making use of complex variables:

Theorem

Be $f_1(x_1, x_2)$ *and* $f_2(x_1, x_2)$ *two scalar plane functions such that*

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0; \tag{23}$$

then a potential function $\Psi(x_1, x_2)$ *exists such that*

$$f_1 = -\frac{\partial \Psi}{\partial x_2}, \quad f_2 = \frac{\partial \Psi}{\partial x_1}. \tag{24}$$

The equilibrium equations of a system subjected to only surface tractions and where $\sigma_{ij} = \sigma_{ij}(x_1, x_2)$ are precisely in the form of (23):

$$
\begin{aligned}
\sigma_{11,1} + \sigma_{12,2} &= 0, \\
\sigma_{21,1} + \sigma_{22,2} &= 0, \qquad \Rightarrow \\
\sigma_{31,1} + \sigma_{32,2} &= 0,
\end{aligned}
\tag{25}
$$

there exist scalar functions $\varphi_i(x_1, x_2)$ such that

$$
\sigma_{i1} = -\varphi_{i,2}, \quad \sigma_{i2} = \varphi_{i,1}. \tag{26}
$$

Because $\sigma_{12} = \sigma_{21}$,

$$
\varphi_{1,1} + \varphi_{2,2} = 0, \tag{27}
$$

which once more is in the form of (23), so it exists a scalar function $\chi(x_1, x_2)$ such that

$$
\varphi_1 = -\chi_{,2}\,, \quad \varphi_2 = \chi_{,1}\,. \tag{28}
$$

Then, by (26), we get the (22).

Putting $\varphi_3 = -\Psi$, called stress function (cf. Ting, 1996), we get also

$$\sigma_{23} = -\Psi_{,1}\,, \quad \sigma_{31} = \Psi_{,2}\,. \tag{29}$$

When the stresses are represented through χ and Ψ by eqs. (22) and (29), then the equilibrium equations are automatically satisfied.

To remark that σ_{33} cannot be determined by this way.

We will see further the use of functions χ and Ψ.

Plane and antiplane states and tensors

The reduction from a 3D to a 2D problem can be done in 2 different cases:

- plane strain
- plane stress

There are substantial differences between the 2 cases, but a common algebraic basis can be given for both of them.

We rewrite the Hooke's law like (p stands for plane and a for antiplane)

$$\begin{cases} \{\sigma^p\} = [C1]\{\varepsilon^p\} + [C2]\{\varepsilon^a\}, \\ \{\sigma^a\} = [C2]^\top\{\varepsilon^p\} + [C3]\{\varepsilon^a\}, \end{cases} \tag{30}$$

and its inverse like

$$\begin{cases} \{\varepsilon^p\} = [S1]\{\sigma^p\} + [S2]\{\sigma^a\}, \\ \{\varepsilon^a\} = [S2]^\top\{\sigma^p\} + [S3]\{\sigma^a\}. \end{cases} \tag{31}$$

In eqs. (30) and (31) it is:

$$\{\sigma^p\} = \left\{ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{array} \right\}, \quad \{\sigma^a\} = \left\{ \begin{array}{c} \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{array} \right\}, \tag{32}$$

$$\{\varepsilon^p\} = \left\{ \begin{array}{c} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{array} \right\}, \quad \{\varepsilon^a\} = \left\{ \begin{array}{c} \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{array} \right\}, \tag{33}$$

$$\begin{gathered} [C1] = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ & C_{22} & C_{26} \\ sym & & C_{66} \end{bmatrix}, \quad [C2] = \begin{bmatrix} C_{13} & C_{14} & C_{15} \\ C_{23} & C_{24} & C_{25} \\ C_{36} & C_{46} & C_{56} \end{bmatrix}, \\ [C3] = \begin{bmatrix} C_{33} & C_{34} & C_{35} \\ & C_{44} & C_{45} \\ sym & & C_{55} \end{bmatrix}, \end{gathered} \tag{34}$$

$$[S1] = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ & S_{22} & S_{26} \\ sym & & S_{66} \end{bmatrix}, \; [S2] = \begin{bmatrix} S_{13} & S_{14} & S_{15} \\ S_{23} & S_{24} & S_{25} \\ S_{36} & S_{46} & S_{56} \end{bmatrix},$$

$$[S3] = \begin{bmatrix} S_{33} & S_{34} & S_{35} \\ & S_{44} & S_{45} \\ sym & & S_{55} \end{bmatrix}. \tag{35}$$

These results are the common algebraic basis for developing, separately but dually, the two cases of plane strain and plane stress.

We will call, in short, plane tensors all those with the superscript p and antiplane all those with the superscript a, i.e. it is antiplane any component out of the plane $x_3 = 0$.

Plane strain

We define plane strain a state for which the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$ is such that

$$u_3 = 0,\ u_\alpha = u_\alpha(x_1, x_2),\ \alpha = 1, 2. \tag{36}$$

Through the strain-displacement relations eq. (36) gives

$$\begin{gathered}\varepsilon_3 = u_{3,3} = 0,\ \varepsilon_4 = \frac{u_{2,3} + u_{3,2}}{2} = 0,\ \varepsilon_5 = \frac{u_{1,3} + u_{3,1}}{2} = 0\ \rightarrow \\ \{\varepsilon^a\} = \{0\}, \quad \{\varepsilon^p\} = \{\varepsilon^p(x_1, x_2)\},\end{gathered} \tag{37}$$

which justifies the name plane strain: the antiplane strain $\{\varepsilon^a\}$ is null and the plane strain $\{\varepsilon^p\}$ is a plane field.

From eqs. (30) and (31) we get hence, for the in plane tensors,

$$\{\sigma^p\} = [C1]\{\varepsilon^p\}, \quad \{\varepsilon^p\} = [\Sigma]\{\sigma^p\}, \tag{38}$$

while for the antiplane tensors it is

$$\begin{aligned} &\{\sigma^a\} = [C2]^\top\{\varepsilon^p\} = -[S3]^{-1}[S2]^\top\{\sigma^p\} = [C2]^\top[\Sigma]\{\sigma^p\},\\ &\{\varepsilon^a\} = \{0\}, \end{aligned} \tag{39}$$

with

$$[\Sigma] = [C1]^{-1} = [S1] - [S2][S3]^{-1}[S2]^\top, \tag{40}$$

the reduced compliance matrix.

The stiffness of the in-plane part, $[C1]$, does not change with respect to the 3D case, while the in-plane compliance $[\Sigma] \neq [S1]$.

Also, unlike $\{\varepsilon^a\}$, $\{\sigma^a\} \neq \{0\}$: the antiplane stress is not null in plane strain

To detail the components of $[\Sigma]$ for a triclinic material is complicate.

Monoclinic material, with $x_3 = 0$ plane of symmetry:

$$\Sigma_{ij} = S_{ij} - \frac{S_{i3}S_{j3}}{S_{33}}, \quad i,j = 1,2,6, \tag{41}$$

and

$$\{\sigma^a\} = \left\{ \begin{array}{c} \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{array} \right\} = \left\{ \begin{array}{c} C_{13}\varepsilon_1 + C_{23}\varepsilon_2 + C_{36}\varepsilon_6 \\ 0 \\ 0 \end{array} \right\}. \tag{42}$$

Through eq. (39) we get also

$$\sigma_3 = -\frac{S_{13}\sigma_1 + S_{23}\sigma_2 + S_{36}\sigma_6}{S_{33}}. \tag{43}$$

$\rightarrow$ The transverse shear components σ_4 and σ_5 vanish in plane strain. This is not the case for σ_3.

Orthotropic material with $\{x_1, x_2, x_3\}$ the orthotropic frame:
because $C_{i6} = S_{i6} = 0 \ \forall i = 1, 2, 3$,

$$[\Sigma] = \begin{bmatrix} S_{11} - \frac{S_{13}^2}{S_{33}} & S_{12} - \frac{S_{13}S_{23}}{S_{33}} & 0 \\ & S_{22} - \frac{S_{23}^2}{S_{33}} & 0 \\ sym & & S_{66} \end{bmatrix}, \tag{44}$$

$$\sigma_3 = C_{13}\varepsilon_1 + C_{23}\varepsilon_2 = -\frac{S_{13}\sigma_1 + S_{23}\sigma_2}{S_{33}}. \tag{45}$$

Isotropic body:

$$[\Sigma] = \begin{bmatrix} \frac{1-\nu^2}{E} & -\frac{\nu(1+\nu)}{E} & 0 \\ & \frac{1-\nu^2}{E} & 0 \\ sym & & \frac{1+\nu}{E} \end{bmatrix}, \tag{46}$$

$$\sigma_3 = \frac{\nu E}{(1-2\nu)(1+\nu)}(\varepsilon_1 + \varepsilon_2) = \nu(\sigma_1 + \sigma_2). \tag{47}$$

Three remarks:

1. condition (36) implies not only that $\{\varepsilon^p\}$ is a plane field, eq. (37), but also, through the Hooke's law, that $\{\sigma\}$ is a plane field too:
$$\sigma_i = \sigma_i(x_1, x_2),\ \forall i = 1, ..., 6; \tag{48}$$
2. plane strain is typical of infinitely long cylindrical bodies subjected to loadings that do not depend upon x_3, the longitudinal axis (e.g. a pipe with internal or/and external pressure, a rail under its own weight etc.). In such cases, the assumption (36) is plausible.
3. generally speaking $\sigma_3(x_1, x_2) \neq 0$. Hence, a plane strain is possible, for finite cylinders, only when appropriate actions are applied at the bases of the cylinder, in order to ensure the existence of $\sigma_3(x_1, x_2) \neq 0$ and that $u_3 = 0$.

The concept of plane strain can get different definitions in the literature; the definition given here, eq. (36), is the same one given by Love, Muskhelishvili and by Rand & Rovenski.

A general and rigorous definition, valid not only for infinitely long cylinders, is given by Milne-Thomson:

> *a state of plane deformation is said to exist if the following conditions are satisfied: (i) one of the principal directions of deformation is the same at every point of the material; (ii) apart from a rigid body movement of the material as a whole, particles which occupy planes perpendicular to the fixed principal direction prior to the deformation continue to occupy the same planes after the deformation.*

To remark the use of the term plane deformation and not of plane strain.

Of course, the above definition implies that there is no warping of the material planes orthogonal to the invariable principal direction and that

$$u_3 = 0, \quad \varepsilon_3 = 0, \tag{49}$$

but not the second assumption of (36), $u_\alpha = u_\alpha(x_1, x_2),\ \alpha = 1, 2$.

Milne-Thomson shows that this is a consequence of the definition of plane strain and of the strain-displacement relation in non-linear elasticity, i.e. taking the Green-Lagrange tensor as measure of deformation

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}) \ \rightarrow \ \varepsilon_3 = u_{3,3} + \frac{1}{2}(u_{1,3}^2 + u_{2,3}^2 + u_{3,3}^2), \tag{50}$$

Through eq. (49), this gives

$$u_{1,3}^2 + u_{2,3}^2 = 0 \;\Rightarrow\; u_{1,3} = u_{2,3} = 0 \;\Rightarrow\; u_\alpha = u_\alpha(x_1, x_2),\;\; \alpha = 1, 2. \tag{51}$$

This result along with $(49)_1$ give

$$\begin{aligned}
\varepsilon_1 &= u_{1,1} + \frac{1}{2}(u_{1,1}^2 + u_{2,1}^2),\\
\varepsilon_2 &= u_{2,2} + \frac{1}{2}(u_{1,2}^2 + u_{2,2}^2),\\
\varepsilon_6 &= \frac{1}{\sqrt{2}}(u_{1,2} + u_{2,1} + u_{1,1}u_{1,2} + u_{2,1}u_{2,2}),\\
\varepsilon_3 &= \varepsilon_4 = \varepsilon_5 = 0.
\end{aligned} \tag{52}$$

Of course, the results of eq. (52) are valid for ε too in the framework of the linearized theory.

By consequence, $\{\sigma\} = \{\sigma(x_1, x_2)\}$ is a plane field too, so giving what Milne-Thomson calls a plane system

Ting introduces the argument as antiplane deformations and then he develops, substantially, the theory described above and later on. He introduces another category of plane deformation problems, those where the only basic assumption is

$$u_i = u_i(x_1, x_2)\ \forall i = 1, 2, 3. \tag{53}$$

There is a substantial difference between this plane case and the one developed above or defined by Milne-Thomson, because now $u_3 \neq 0$. Ting calls this type of plane deformation the Stroh formalism

Green & Zerna introduce the concept of plane strain as a system where the displacement and strain components are independent from x_3, so substantially the same definition given by Ting for the Stroh formalism, and develop all the theory in the framework of nonlinear elasticity, which is far beyond our scope.

Plane stress

An elastic body is in a plane stress state when the antiplane stress $\{\sigma^a\}$ is null and the plane stress $\{\sigma^p\}$ is a plane function:

$$\begin{aligned}
&\{\sigma^a\} = \{0\} \;\rightarrow\; \sigma_3 = \sigma_4 = \sigma_5 = 0, \\
&\{\sigma^p\} = \{\sigma^p(x_1, x_2)\}, \;\rightarrow\; \sigma_i = \sigma_i(x_1, x_2) \;\forall i = 1, 2, 6.
\end{aligned} \tag{54}$$

As a consequence of (54) and of the equation of motion also the body vector is a plane function: $\mathbf{b} = \mathbf{b}(x_1, x_2)$.

The case of plane stress is completely analogous to the previous one of plane strain: because of the symmetry of relations (30) and (31), the developments for plane stress can be obtained repeating *verbatim* those for plane strain, simply replacing the strains with stresses, the compliances with stiffnesses:

$$\begin{aligned}
\{\sigma^p\} &= [Q]\{\varepsilon^p\}, \\
\{\varepsilon^p\} &= [S1]\{\sigma^p\},
\end{aligned} \tag{55}$$

and for the antiplane tensors

$$
\begin{aligned}
&\{\sigma^a\} = 0, \\
&\{\varepsilon^a\} = [S2]^\top\{\sigma^p\} = -[C3]^{-1}[C2]^\top\{\varepsilon^p\} = [S2]^\top[Q]\{\varepsilon^p\},
\end{aligned} \tag{56}
$$

with

$$
[Q] = [S1]^{-1} = [C1] - [C2][C3]^{-1}[C2]^\top, \tag{57}
$$

the reduced stiffness matrix.

In a dual manner with respect to the results of plane strain, in case of plane stress the compliance of the in-plane part, $[S1]$, does not change with respect to the 3D case, while the in-plane stiffness changes: $[Q] \neq [C1]$.

Also, unlike $\{\sigma^a\}$, $\{\varepsilon^a\} \neq \{0\}$: the antiplane strain is not null in plane stress, generally speaking.

For a monoclinic material we obtain

$$Q_{ij} = C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}}, \quad i,j = 1,2,6, \tag{58}$$

and

$$\{\varepsilon^a\} = \left\{ \begin{array}{c} \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{array} \right\} = \left\{ \begin{array}{c} S_{13}\sigma_1 + S_{23}\sigma_2 + S_{36}\sigma_6 \\ 0 \\ 0 \end{array} \right\}. \tag{59}$$

Through eq. (56) we get also

$$\varepsilon_3 = -\frac{C_{13}\varepsilon_1 + C_{23}\varepsilon_2 + C_{36}\varepsilon_6}{C_{33}}. \tag{60}$$

So, in the case of monoclinic material with $x_3 = 0$ plane of symmetry, the transverse shear deformations ε_4 and ε_5 vanish in plane stress, but not ε_3.

For an orthotropic material with $\{x_1, x_2, x_3\}$ the orthotropic frame we get:

$$[Q] = \begin{bmatrix} C_{11} - \frac{C_{13}^2}{C_{33}} & C_{12} - \frac{C_{13}C_{23}}{C_{33}} & 0 \\ & C_{22} - \frac{C_{23}^2}{C_{33}} & 0 \\ sym & & C_{66} \end{bmatrix}, \tag{61}$$

$$\varepsilon_3 = S_{13}\sigma_1 + S_{23}\sigma_2 = -\frac{C_{13}\varepsilon_1 + C_{23}\varepsilon_2}{C_{33}}. \tag{62}$$

Using the fact that $[Q] = [S1]^{-1}$, we get also

$$[Q] = \begin{bmatrix} \frac{S_{22}}{S_{11}S_{22}-S_{12}^2} & -\frac{S_{12}}{S_{11}S_{22}-S_{12}^2} & 0 \\ & \frac{S_{11}}{S_{11}S_{22}-S_{12}^2} & 0 \\ sym & & \frac{1}{S_{66}} \end{bmatrix}; \tag{63}$$

this result gives a bound on the Young's moduli: because

$$S_{ii} = \frac{1}{E_i}, \quad S_{ji} = -\frac{\nu_{ij}}{E_i}, \quad \frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \Rightarrow Q_{ii} > E_i, \ \ i = 1, 2 \tag{64}$$

Finally, for an isotropic body we get

$$[Q] = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ & \frac{E}{1-\nu^2} & 0 \\ sym & & \frac{E}{1+\nu} \end{bmatrix}, \tag{65}$$

$$\varepsilon_3 = -\frac{\nu}{E}(\sigma_1 + \sigma_2) = -\frac{\nu}{1-\nu}(\varepsilon_1 + \varepsilon_2). \tag{66}$$

A remark about the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$: generally speaking, it is not a plane function:

$$\mathbf{u} = \mathbf{u}(x_1, x_2, x_3), \tag{67}$$

i.e., the problem is not plane for the displacements. This is a fundamental difference with plane strain; in fact, for plane strain, $\mathbf{u}$, ε and $\boldsymbol{\sigma}$ are all plane fields.

To end this section, some commentary about the notion of plane stress in the literature. The definition given here, eq. (54) is rather classical, and it is, for instance, that given by Love or by Rand & Rovenski.

Milne-Thomson gives perhaps the most general definition:

> *a plane system is one for which there exists a plane such that the stress tensor is the same at all material points of any normal to this plane as at the material point in which that normal meets the plane.*

To remark the use of the term plane system and not of plane stress by Milne -Thomson. Also, his definition is not completely identical to that given in (54), because it is not required that condition $(54)_1$ be satisfied.

Nevertheless, the same author immediately after considers only plane systems with $\sigma_4 = \sigma_5 = 0$. This implies that, for the third equation of motion, the body vector **b** is planar: $b_3 = 0$. So, all the actions are parallel to the plane of the system.

Lekhnitskii analyzes exactly the general case of plane system as defined by Milne-Thomson and Ting ting calls explicitly such a system the Lekhnitskii Formalism.

The state of plane stress is typical of thin, flat bodies, like plates or slabs. A plate is thin when its thickness is much smaller than its typical in-plane dimension.

If the plate is submitted to only in-plane loadings, then, because of the small thickness of the plate and assuming a continuous distribution of the σ_{ij}s through the plate's thickness, assumptions (54) are a good approximation of reality.

Generalized plane stress

The concept of generalized plane stress was first introduced by Filon (1903) and successively developed by Love, Muskhelishvili and by Lekhnitskii, as a special case of his plane theory.

Let us consider a thin plate whose thickness is $2t$, acted upon only by loadings parallel to the mid-plane $x_3 = 0$ and with the two surfaces unloaded:

$$\sigma_3 = \sigma_4 = \sigma_5 = 0 \quad \text{at} \quad x_3 = \pm t. \tag{68}$$

For a triclinic material, the plane stress $\{\sigma^p\}$ will generate also antiplane strains, $\{\varepsilon^a\} \neq \{0\}$, which implies that $u_3(x_1, x_2, 0) \neq 0$: the mid-plane of the plate will warp under in-plane loadings.

To exclude this possibility, we will consider only anisotropic materials with at least

$$C_{14} = C_{15} = C_{24} = C_{25} = C_{34} = C_{35} = C_{46} = C_{56} = 0 \tag{69}$$

The most general materials satisfying such requirements, are those of the monoclinic syngony with $x_3 = 0$ as plane of symmetry.

We introduce the average displacements

$$\widehat{u}_i = \frac{1}{2t} \int_{-t}^{+t} u_i \; dx_3 \; \forall i = 1, 2, 3, \tag{70}$$

and

$$[u_i] = \frac{u_i(x_1, x_2, t) - u_i(x_1, x_2, -t)}{2t}. \tag{71}$$

We make the further assumption that all the applied forces are symmetrically distributed with respect to the mid-plane of the plate, so that the stresses are symmetric with respect to this plane.

As a consequence, also the displacements will be symmetric and, by (70) and (71), it will be

$$[u_1] = [u_2] = 0, \quad \widehat{u}_3 = 0 \tag{72}$$

which gives

$$\frac{1}{2t}\int_{-t}^{+t} u_{i,j}\ dx_3 = \begin{cases} \widehat{u}_{i,j} \quad \forall i,j \neq 3, \\ \widehat{\varepsilon}_3 \ \ \text{if } i=j=3, \\ 0 \ \ \text{otherwise.} \end{cases} \tag{73}$$

This result means that the average displacement is a plane vector and also a plane field:

$$\widehat{u}_3 = 0, \quad \widehat{u}_\alpha = \widehat{u}_\alpha(x_1, x_2),\ \alpha = 1, 2, \tag{74}$$

and that for the average strain it is

$$\widehat{\varepsilon}_4 = \widehat{\varepsilon}_5 = 0, \quad \widehat{\varepsilon}_3 \neq 0, \quad \{\widehat{\varepsilon}\} = \{\widehat{\varepsilon}(x_1, x_2)\}, \tag{75}$$

i.e. the strain tensor in not plane but it is a plane field.

As a consequence, considering the requirements (69), integrating the Hooke's law over the thickness gives

$$\widehat{\sigma}_i = C_{i1}\widehat{\varepsilon}_1 + C_{i2}\widehat{\varepsilon}_2 + C_{i3}\widehat{\varepsilon}_3 + C_{i6}\widehat{\varepsilon}_6. \tag{76}$$

Applying the third equilibrium equation

$$\sigma_{5,1} + \sigma_{4,2} + \sqrt{2}\sigma_{3,3} = 0, \tag{77}$$

at the plate's surfaces, $x_3 = \pm t$, for the (68) we get

$$\sigma_{3,3} = 0. \tag{78}$$

The consequence of (68) and of the last result is

$$\sigma_3 \simeq 0 \quad \forall x_3 \in [-t, t] \quad \Rightarrow \quad \widehat{\sigma}_3 = 0. \tag{79}$$

Then, writing the (76) for $\widehat{\sigma}_{33}$ gives the condition

$$\widehat{\varepsilon}_3 = -\frac{1}{C_{33}} \left(C_{31}\widehat{\varepsilon}_1 + C_{32}\widehat{\varepsilon}_2 + C_{36}\widehat{\varepsilon}_6 \right), \tag{80}$$

that injected back into (76) gives

$$\{\widehat{\sigma}\} = [\widehat{C}]\{\widehat{\varepsilon}\}, \tag{81}$$

with $[\widehat{C}]$ the reduced elastic stiffness matrix:

$$\widehat{C}_{ij} = C_{ij} - \frac{C_{i3}C_{j33}}{C_{33}}. \tag{82}$$

The above components define the reduced elastic stiffness matrix exactly as $[Q]$, see eq. (58).

Nevertheless, the difference with plane stress is that in generalized plane stress all the equations are satisfied on the average.

If t is very small compared to the other relevant dimensions of the plate then generalized plane stress is a good approximation.

To notice that, through eqs. (69) and (75) it is

$$\widehat{\sigma}_4 = \widehat{\sigma}_5 = 0. \tag{83}$$

Let us now integrate the equilibrium equations on the thickness of the plate; then, eq. (68) gives

$$\begin{aligned} &\sqrt{2}\widehat{\sigma}_{1,1} + \widehat{\sigma}_{6,2} = 0, \\ &\widehat{\sigma}_{6,1} + \sqrt{2}\widehat{\sigma}_{2,2} = 0, \\ &\widehat{\sigma}_{5,1} + \widehat{\sigma}_{4,2} = 0. \end{aligned} \tag{84}$$

Mechanical consistency of plane states

We have introduced plane strain and plane stress; we ponder now their mechanical consistency, i.e. if such states are physically possible.

Plane strain: injecting the Hooke's law in the equilibrium equations of a body submitted only to loadings on its boundary, these reduce to

$$\begin{aligned}&E_{i111}\varepsilon_{11,1}+2E_{i112}\varepsilon_{12,1}+E_{i122}\varepsilon_{22,1}+\\&E_{i211}\varepsilon_{11,2}+2E_{i212}\varepsilon_{12,2}+E_{i222}\varepsilon_{22,2}=0\end{aligned}\quad\forall i=1,2,3. \tag{85}$$

The coefficients of the third equation are

$$\begin{aligned}&E_{3111}=C_{15},E_{3121}=E_{3112}=C_{56},E_{3212}=E_{3221}=C_{46},\\&E_{3222}=C_{24},E_{3211}=C_{14},E_{3122}=C_{25}.\end{aligned} \tag{86}$$

All of these coefficients are null for a monoclinic material with $x_3=0$ as plane of symmetry.

Let us now consider the antiplane deformations

$$u_1 = u_2 = 0, \quad u_3 = u_3(x_1, x_2) \;\rightarrow\; u_{3,3} = \varepsilon_{33} = 0, \;\{\varepsilon^p\} = \{0\}. \tag{87}$$

Now the three equations of equilibrium reduce to

$$E_{i113}\varepsilon_{13,1} + E_{i123}\varepsilon_{23,1} + E_{i213}\varepsilon_{13,2} + E_{i223}\varepsilon_{23,2} = 0 \;\forall i = 1, 2, 3. \tag{88}$$

The coefficients of the two first equations (88) are exactly the (86).

Hence, a monoclinic body satisfies automatically, for each applied loading on the boundary, the third plane equilibrium equation and the two first antiplane equations: plane and antiplane deformations are uncoupled.

The monoclinic condition is not the minimal requirement: the true necessary conditions are the (86) to be null, while for a monoclinic material, required for generalized plane stress, it is also $C_{34} = C_{35} = 0$

This result is obviously valid also for the other elastic syngonies that satisfy the same conditions, namely for the orthotropic, tetragonal, axially-symmetric, cubic and isotropic ones.

For all such materials, the plane strain state is a possible situation and it is an exact theory. To remark that in this circumstance, it is also $\sigma_4 = \sigma_5 = 0$.

For a triclinic or trigonal body, or for any other syngony not correctly oriented (i.e. for which $x_3 = 0$ is not one of the symmetry planes), a plane strain deformation or an antiplane one cannot exist, generally speaking: also in the case where the three components of displacement u_i depend upon only x_1 and x_2, all of them are coupled, so that u_3 does not vanish, in general.

Such a state is called a generalized plane strain: $u_3 \neq 0$, but $\varepsilon_3 = 0$ because nothing is function of x_3, so that $u_{3,3} = 0$.

The compatibility equations give an equation for the Airy's stress function (22).

In fact, with the assumptions (37) all the compatibility equations are automatically satisfied but the first

$$\sqrt{2}\varepsilon_{6,12} = \varepsilon_{1,22} + \varepsilon_{2,11}. \tag{89}$$

Using eq. $(38)_2$ and expressing the stress components by the (22), remembering that $\sigma_6 = \sqrt{2}\sigma_{12}$, we get the following homogenized biharmonic equation for the Airy's stress function χ:

$$\nabla_1^4\chi = 0, \tag{90}$$

where

$$\begin{aligned}\nabla_1^4 = \varSigma_{22}\frac{\partial^4}{\partial x_1^4} - 2\sqrt{2}\varSigma_{26}\frac{\partial^4}{\partial x_1^3\partial x_2} + 2(\varSigma_{12} + \varSigma_{66})\frac{\partial^4}{\partial x_1^2\partial x_2^2} - \\ 2\sqrt{2}\varSigma_{16}\frac{\partial^4}{\partial x_1\partial x_2^3} + \varSigma_{11}\frac{\partial^4}{\partial x_2^4}\end{aligned} \tag{91}$$

is the generalized biharmonic differential operator.

Changing of material syngony or of plane state, other operators can be introduced; for instance, it can be easily checked that for an orthotropic material, $\Sigma_{16} = \Sigma_{26} = 0$, so that ∇_1^4 has a simpler form, while for an isotropic material we get $\nabla_1^4 = \frac{1-\nu^2}{E}\nabla^4$, where ∇^4 is the customary double laplacian.

Plane stress: if the Airy's function is used in the first compatibility equation and proceeding like in the previous case, but now with the strain-stress relation $(55)_2$, we get the biharmonic equation for χ

$$\nabla_2^4\chi = 0, \tag{92}$$

with now

$$\begin{aligned}\nabla_2^4 = S_{22}\frac{\partial^4}{\partial x_1^4} - 2\sqrt{2}S_{26}\frac{\partial^4}{\partial x_1^3\partial x_2} + 2(S_{12}+S_{66})\frac{\partial^4}{\partial x_1^2\partial x_2^2} - \\ 2\sqrt{2}S_{16}\frac{\partial^4}{\partial x_1\partial x_2^3} + S_{11}\frac{\partial^4}{\partial x_2^4}\end{aligned} \tag{93}$$

the generalized biharmonic operator for the present case.

Formally, ∇_2^4 is identical to ∇_1^4, but the components of the compliance tensor $[S]$ are to be used in place of those of the reduced compliance $[\Sigma]$.

The other compatibility equations, for a strain tensor that is a plane field but not a plane tensor, because generally speaking in plane stress $\{\varepsilon^a\} \neq \{0\}$, are

$$\varepsilon_{3,11} = 0, \;\; \varepsilon_{3,12} = 0, \;\; \varepsilon_{3,22} = 0, \;\; \varepsilon_{4,11} = \varepsilon_{5,12}, \;\; \varepsilon_{4,12} = \varepsilon_{5,22}. \quad (94)$$

Also considering materials that are at least monoclinic, for which $\varepsilon_4 = \varepsilon_5 = 0$, so that the two last equations are automatically satisfied, the first three equations are left unsatisfied, unless ε_3 is a linear function of x_1, x_2:

$$\varepsilon_3(x_1, x_2) = c_0 + c_1 x_1 + c_2 x_2. \quad (95)$$

In all the other cases, the plane stress analysis is not exact, and can be considered as accurate only in the limit of thin plates acted upon by surface tractions parallel to the mid-plane of the plate.

Comparison of plane states

Plane strain:

- the displacement is a plane vector and also a plane field: $u_3 = 0,\ u_\alpha = u_\alpha(x_1, x_2), \alpha = 1, 2;$
- the strain tensor is plane and also a plane field: $\{\varepsilon\} = \{\varepsilon^p(x_1, x_2)\},\ \ \{\varepsilon^a\} = \{0\};$
- the stress tensor is not plane but it is a plane field: $\{\sigma\} = \{\sigma(x_1, x_2)\}$; for a material with the moduli (86) null, it is also $\sigma_4 = \sigma_5 = 0$, but $\sigma_3 \neq 0$;
- the equilibrium equations in case of null body vector, for a material with the moduli (86) null, reduce to $\sigma_{ij,j} = 0\ \ j = 1, 2, \forall i = 1, 2, 3$; the third equation corresponds to the antiplane state, uncoupled from the plane one;
- the Hooke's law does not change with respect to the 3D case: $\{\sigma\} = [C]\{\varepsilon^p\};$
- the inverse Hooke's law becomes: $\{\varepsilon^p\} = [\Sigma]\{\sigma^p\}$, with $[\Sigma]$ the reduced compliance matrix whose components are given by eq. (41) for a material at least monoclinic;
- the theory of plane strain is exact.

Plane stress

- the displacement is not a plane vector nor a plane field: $u_i = u_i(x_1, x_2, x_3), \ \forall i = 1, 2, 3$;
- the strain tensor is not plane but it is a plane field: $\{\varepsilon\} = \{\varepsilon(x_1, x_2)\}$; for a material at least monoclinic, it is also $\varepsilon_4 = \varepsilon_5 = 0$, but $\varepsilon_3 \neq 0$;
- the stress tensor is plane and also a plane field: $\{\sigma\} = \{\sigma^p(x_1, x_2)\}$;
- the equilibrium equations for a null body vector reduce to $\sigma_{ij,j} = 0 \ \ i, j = 1, 2$, regardless of the material;
- the Hooke's law becomes: $\{\sigma^p\} = [Q]\{\varepsilon^p\}$, with $[Q]$ the reduced stiffness matrix whose components for a material at least monoclinic are given by eq. (58);
- the inverse Hooke's law does not change with respect to the 3D case: $\{\varepsilon\} = [S]\{\sigma^p\}$;
- the theory of plane stress is not exact.

Generalized plane stress

- all the relations are given on the average, i.e. as average values on the thickness of the plate, not locally;
- the theory is valid for thin plates of a material at least monoclinic, with $\sigma_3 = \sigma_4 = \sigma_5 = 0$ on the plate's surfaces and submitted uniquely to loadings parallel to the plate's mid-plane;
- the average displacement is a plane vector and also a plane field: $\widehat{u}_3 = 0, \quad \widehat{u}_\alpha = \widehat{u}_\alpha(x_1, x_2), \; \alpha = 1, 2;$
- the average strain tensor is not plane, because $\widehat{\varepsilon}_4 = \widehat{\varepsilon}_5 = 0$, but $\widehat{\varepsilon}_3 \neq 0$; nevertheless, it is a plane field: $\{\widehat{\varepsilon}\} = \{\widehat{\varepsilon}(x_1, x_2)\}$;
- the average stress tensor is not plane but it is a plane field: $\{\widehat{\sigma}\} = \{\widehat{\sigma}^p(x_1, x_2)\}$;
- the equilibrium equations reduce to $\widehat{\sigma}_{ij,j} = 0 \quad j = 1, 2 \; \forall i = 1, 2, 3;$

- the average Hooke's law becomes: $\{\widehat{\sigma}\} = [\widehat{C}]\{\widehat{\varepsilon}\}$, with $[\widehat{C}] = [Q]$ the reduced stiffness matrix whose components are given by eq. (58);
- the theory of generalized plane stress is exact, on the average, only if $\widehat{\sigma}_3$ is exactly zero everywhere in the plate.

It appears hence that plane strain and generalized plane stress are formally identical, provided that the stiffness matrix of plane strain is replaced by the reduced stiffness matrix for generalized plane stress, and of course considering that in generalized plane stress all the relations are valid on the average.

Nevertheless, some differences remain, for instance $\sigma_3 \neq 0$ and $\varepsilon_3 = 0$ in plane strain, while it is assumed that $\widehat{\sigma}_3 = 0$ and $\varepsilon_3 \neq 0$ in generalized plane stress.

The case of plane stress is not formally identical to plane strain nor to generalized plane stress because the displacement vector is not plane nor a plane field, besides the fact that $\varepsilon_3 \neq 0$ and $\sigma_3 = 0$.

The Lekhnitskii theory

We will name a Lekhnitskii Problem every problem of the elastic equilibrium of an anisotropic body whose stress field is constrained to satisfy uniquely the condition of plane field:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(x_1, x_2), \tag{96}$$

hence, generally speaking, with $\{\sigma^a\} \neq \{0\}$, i.e. the stress is not necessarily a plane tensor.

The same properties, by the reverse Hooke's law, are true for the strain too, but not for the displacement:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(x_1, x_2), \quad \mathbf{u} = \mathbf{u}(x_1, x_2, x_3). \tag{97}$$

The Lekhnitskii theory or formalism is the mathematical theory for obtaining a general formulation of the solution to the Lekhnitskii Problem.

It is based upon the use of the stress functions χ and Ψ. We will see that the Lekhnitskii theory comprehends, as special cases:

- plane deformation
- generalized plane strain
- generalized plane stress

We follow here the original approach of Lekhnitskii, considering the general case of an anisotropic body belonging to any possible elastic syngony, submitted to surface tractions on the boundaries and to volume forces depending upon a potential U,

$$\mathbf{f} = \nabla U. \tag{98}$$

The decomposition of the displacement field

The displacement vector $\mathbf{u}(x_1, x_2, x_3)$ is decomposed into a plane vector field[1]

$$\mathbf{u}^p = \mathbf{u}^p(x_1, x_2) \tag{99}$$

and a field complementary to the plane one, depending also upon x_3. This can be done in the following way: in the Kelvin's notation, it is

$$\begin{aligned}
&\varepsilon_1 = u_{1,1}, \quad \varepsilon_2 = u_{2,2}, \quad \varepsilon_3 = u_{3,3}, \\
&\varepsilon_4 = \frac{u_{2,3} + u_{3,2}}{\sqrt{2}}, \quad \varepsilon_5 = \frac{u_{1,3} + u_{3,1}}{\sqrt{2}}, \quad \varepsilon_6 = \frac{u_{1,2} + u_{2,1}}{\sqrt{2}}.
\end{aligned} \tag{100}$$

Because the stress is a plane field, for the Hooke's law it is also

$$\varepsilon_i = \varepsilon_i(x_1, x_2) \quad \forall i = 1, ..., 6; \tag{101}$$

hence in eq. (100) the right-hand sides are independent of x_3.

[1] Here the symbol p denotes only a plane field, not a plane vector.

The most general expression for the components of $\mathbf{u}(x_1, x_2, x_3)$ is

$$
\begin{aligned}
u_1(x_1, x_2, x_3) &= u_1^p(x_1, x_2) + u(x_2, x_3), \\
u_2(x_1, x_2, x_3) &= u_2^p(x_1, x_2) + v(x_1, x_3), \\
u_3(x_1, x_2, x_3) &= u_3^p(x_1, x_2) + x_3 w(x_1, x_2).
\end{aligned}
\tag{102}
$$

Injecting eq. (102) into eq. $(100)_{4,5,6}$ gives

$$
\begin{aligned}
&\varepsilon_4 = \frac{u^p_{3,2} + v_{,3} + x_3 w_{,2}}{\sqrt{2}}, \quad \varepsilon_5 = \frac{u^p_{3,1} + u_{,3} + x_3 w_{,1}}{\sqrt{2}}, \\
&\varepsilon_6 = \frac{u^p_{1,2} + u^p_{2,1} + u_{,2} + v_{,1}}{\sqrt{2}},
\end{aligned}
\tag{103}
$$

and because of eqs. (99) and (101), the quantities

$$
v_{,3} + x_3 w_{,2}, \quad u_{,3} + x_3 w_{,1}, \quad u_{,2} + v_{,1} \tag{104}
$$

cannot depend upon x_3.

Then, $u_{,3}$ and $v_{,3}$ must be linear in x_3, while $w_{,1}$ is a function of x_1 and $w_{,2}$ of x_2. Then, we can put

$$\begin{aligned} u(x_2,x_3) &= -\frac{1}{2}x_3^2(A+D\ x_2)+x_3 f(x_2),\\ v(x_1,x_3) &= -\frac{1}{2}x_3^2(B+D\ x_1)+x_3 g(x_1),\\ w(x_1,x_2) &= A\ x_1+B\ x_2+C+D\ x_1x_2,\quad A,B,C,D\in\mathbb{R}. \end{aligned} \tag{105}$$

Injecting (105) into $(104)_3$ leads to

$$-D\ x_3^2+x_3\frac{df(x_2)}{dx_2}+x_3\frac{dg(x_1)}{dx_1}, \tag{106}$$

a quantity that must be independent of x_3, which gives

$$D=0,\quad f(x_2)=-(\omega x_2+\gamma_2),\quad g(x_1)=\omega x_1+\gamma_1,\quad \omega,\gamma_1,\gamma_2\in\mathbb{R}. \tag{107}$$

Hence, the displacement field has the expression

$$\begin{aligned} u_1(x_1,x_2,x_3) &= u_1^p(x_1,x_2) - \frac{1}{2}A\, x_3^2 - \omega x_2 x_3 - \gamma_2 x_3, \\ u_2(x_1,x_2,x_3) &= u_2^p(x_1,x_2) - \frac{1}{2}B\, x_3^2 + \omega x_1 x_3 + \gamma_1 x_3, \\ u_3(x_1,x_2,x_3) &= u_3^p(x_1,x_2) + x_3(A\, x_1 + B\, x_2 + C). \end{aligned} \tag{108}$$

Any rigid displacement can be added to $\mathbf{u}(x_1,x_2,x_3)$ without altering the strain and stress fields; we can hence add the displacement $\boldsymbol{\delta}(x_1,x_2,x_3)$ corresponding to an infinitesimal rigid rotation θ around the axis $\boldsymbol{\gamma}=(\gamma_1,\gamma_2,\gamma_3)$,

$$\mathbf{R}_\gamma = \mathbf{I} + \boldsymbol{\Gamma}, \tag{109}$$

with $\boldsymbol{\Gamma}$ the axial tensor corresponding to $\boldsymbol{\gamma}$:

$$\boldsymbol{\Gamma} = \begin{bmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{bmatrix}; \tag{110}$$

hence

$$\delta(x_1, x_2, x_3) = \mathbf{R}_\gamma \mathbf{x} - \mathbf{x} = \left\{ \begin{array}{c} \gamma_2 x_3 - \gamma_3 x_2 \\ \gamma_3 x_1 - \gamma_1 x_3 \\ \gamma_1 x_2 - \gamma_2 x_1 \end{array} \right\}. \tag{111}$$

Once (111) added to (108) and the terms depending upon x_1 and x_2 incorporated in the $u_i^p(x_1, x_2)$, we get

$$\begin{aligned} u_1(x_1, x_2, x_3) &= u_1^p(x_1, x_2) - \frac{1}{2} A\ x_3^2 - \omega x_2 x_3, \\ u_2(x_1, x_2, x_3) &= u_2^p(x_1, x_2) - \frac{1}{2} B\ x_3^2 + \omega x_1 x_3, \\ u_3(x_1, x_2, x_3) &= u_3^p(x_1, x_2) + x_3(A\ x_1 + B\ x_2 + C). \end{aligned} \tag{112}$$

The terms in (112) depending upon x_3 account for the difference between plane stress field or plane displacement field (but not of plane strain, when the assumption $u_3 = 0$ is also done), and finally between the Lekhnitskii and the Stroh theories.

Strain field and compatibility equations

With the components (112), eq. (100) becomes

$$\begin{aligned}
&\varepsilon_1 = u^p_{1,1}, \quad \varepsilon_2 = u^p_{2,2}, \quad \varepsilon_3 = A\ x_1 + B\ x_2 + C, \\
&\varepsilon_4 = \frac{u^p_{3,2} + \omega x_1}{\sqrt{2}}, \quad \varepsilon_5 = \frac{u^p_{3,1} - \omega x_2}{\sqrt{2}}, \quad \varepsilon_6 = \frac{u^p_{1,2} + u^p_{2,1}}{\sqrt{2}}.
\end{aligned} \tag{113}$$

ε_3 is linear in x_1 and x_2 $\Rightarrow$ the deformation corresponds to a bending about the line $A\ x_1 + B\ x_2 + C = 0$.

The deformation determined by ω is a torsion about the axis of x_3.

With these ε_is the only compatibility equations that are not identically satisfied are

$$\begin{aligned}
&\varepsilon_{1,22} + \varepsilon_{2,11} = \sqrt{2}\ \varepsilon_{6,12}, \\
&\varepsilon_{4,1} - \varepsilon_{5,2} = \sqrt{2}\ \omega.
\end{aligned} \tag{114}$$

These relations will give the two differential equations to be satisfied by the stress functions χ and Ψ.

Differential equations for χ and Ψ

χ and Ψ cannot determine $\sigma_3 \Rightarrow$ if a solution is looked for in terms of χ and Ψ, σ_3 must be eliminated.

This can be done deducing σ_3 from

$$\varepsilon_i = S_{ij}\sigma_j \;\rightarrow\; \sigma_3 = \frac{\varepsilon_3}{S_{33}} - \frac{1}{S_{33}} \sum_{\substack{j=1 \\ j\neq 3}}^{6} S_{3j}\sigma_j, \tag{115}$$

which injected back into the Hooke's reverse law gives

$$\varepsilon_i = S'_{i1}\sigma_1 + S'_{i2}\sigma_2 + S'_{i4}\sigma_4 + S'_{i5}\sigma_5 + S'_{i6}\sigma_6 + S^*_{i3}\varepsilon_3, \quad i = 1, 2, 4, 5, 6. \tag{116}$$

with

$$S'_{ij} = S_{ij} - \frac{S_{i3}S_{j3}}{S_{33}}, \quad S^*_{i3} = \frac{S_{i3}}{S_{33}}, \quad i, j = 1, ..., 6. \tag{117}$$

The components S'_{ij} are called reduced elastic compliances, and they are exactly equal to the components Σ_{ij}, also called reduced compliances.

This is rather surprisingly, because the Σ_{ij}s arise in a plane strain problem, quite different from the Lekhnitskii theory, where the only assumption is a plane field for stress.

Actually, there are important differences between the S'_{ij}s and the Σ_{ij}s: while the S'_{ij}s are valid for each elastic syngony, the Σ_{ij}s are correct only for a material at least monoclinic with $x_3 = 0$ as plane of symmetry.

Moreover, the S'_{ij}s are defined for the 3D case, while the Σ_{ij}s define only plane components.

Actually, though the S'_{ij}s are equal to the Σ_{ij}s, they are deduced in a completely different way, which explains why in a problem with a plane stress field, which however is not a plane stress state, there are reduced compliances and not reduced stiffnesses.

To remark that, with definition (117),

$$S'_{ij} = S'_{ji}, \tag{118}$$

and

$$S'_{i3} = S'_{3i} = 0 \quad \forall i = 1, ..., 6. \tag{119}$$

We now express the σ_{ij}s using the stress functions χ and Ψ,

$$\begin{aligned}
\sigma_1 &= \chi_{,22} - U, \\
\sigma_2 &= \chi_{,11} - U, \\
\sigma_6 &= -\sqrt{2}\,\chi_{,12}, \\
\sigma_4 &= -\sqrt{2}\,\Psi_{,1}, \\
\sigma_5 &= \sqrt{2}\,\Psi_{,2}.
\end{aligned} \tag{120}$$

Substituting these relations into eq. (116) gives

$$\varepsilon_i = S'_{i1}(\chi_{22} - U) + S'_{i2}(\chi_{11} - U) - \sqrt{2}S'_{i4}\Psi_{,1} + \sqrt{2}S'_{i5}\Psi_{,2} - \sqrt{2}S'_{i6}\chi_{,12} + S^*_{i3}\varepsilon_3, \quad i = 1, 2, 4, 5, 6. \tag{121}$$

The derivatives of the ε_is can now be calculated and injected into the compatibility equations (114); remembering the expression of ε_3, eq. $(113)_3$, some standard passages lead to the following result:

$$\begin{aligned} \nabla_1^4\chi + \nabla_1^3\Psi &= C_1, \\ \nabla_1^3\chi + \nabla_1^2\Psi &= C_2, \end{aligned} \tag{122}$$

where the known terms at the right-hand side C_1 and C_2 are

$$\begin{aligned} C_1 &= (S'_{12} + S'_{22})U_{,11} - \sqrt{2}(S'_{16} + S'_{26})U_{,12} + (S'_{11} + S'_{12})U_{,22}, \\ C_2 &= -2\omega + \sqrt{2}\left[S^*_{34}A - S^*_{35}B - (S'_{14} + S'_{24})U_{,1} + (S'_{15} + S'_{25})U_{,2}\right]. \end{aligned} \tag{123}$$

The differential operators are

$$
\begin{aligned}
\nabla_1^2 &= 2\left(S'_{44}\frac{\partial^2}{\partial x_1^2} - 2S'_{45}\frac{\partial^2}{\partial x_1 \partial x_2} + S'_{55}\frac{\partial^2}{\partial x_2^2}\right), \\
\nabla_1^3 &= \sqrt{2}\left[-S'_{24}\frac{\partial^3}{\partial x_1^3} + (S'_{25} + \sqrt{2}S'_{46})\frac{\partial^3}{\partial x_1^2 \partial x_2} - \right. \\
&\left. (S'_{14} + \sqrt{2}S'_{56})\frac{\partial^3}{\partial x_1 \partial x_2^2} + S'_{15}\frac{\partial^3}{\partial x_2^3}\right], \\
\nabla_1^4 &= S'_{22}\frac{\partial^4}{\partial x_1^4} - 2\sqrt{2}S'_{26}\frac{\partial^4}{\partial x_1^3 \partial x_2} + 2(S'_{12} + S'_{66})\frac{\partial^4}{\partial x_1^2 \partial x_2^2} - \\
&2\sqrt{2}S'_{16}\frac{\partial^4}{\partial x_1 \partial x_2^3} + S'_{11}\frac{\partial^4}{\partial x_2^4}.
\end{aligned}
\tag{124}
$$

∇_1^4 is not only formally identical to the generalized biharmonic operator of the plane strain state but, because of the above mentioned identity of the S_{ij}s and Σ_{ij}s, they are exactly the same operator; that is why we have indicated with the same symbol both of them.

Equations (122) are a system of non-homogeneous differential equations for χ and Ψ; together with the appropriate boundary conditions, they define a boundary value problem reduced to the knowledge of the scalar two-dimensional functions χ and Ψ.

The Lekhnitskii theory has hence transformed a 3D problem into a two-dimensional one, the dependence upon x_3 being however recovered in the above relations for the ε_i and $\mathbf{u}$.

The equations in (122) can be rearranged for uncoupling χ and Ψ and for obtaining a homogeneous problem. To this end, let us pose

$$\chi = \chi^h + \chi^p, \quad \Psi = \Psi^h + \Psi^p, \tag{125}$$

h: solutions of the associated homogeneous equations:

$$\nabla_1^4 \chi^h + \nabla_1^3 \Psi^h = 0, \quad \nabla_1^3 \chi^h + \nabla_1^2 \Psi^h = 0, \tag{126}$$

p: particular solution of eq. (122) depending upon the known terms (123) and usually rather simple to be found.

Homogeneous equations (126): we uncouple χ and Ψ:

$$\begin{array}{rl} \nabla_1^2(\nabla_1^4\chi^h + \nabla_1^3\Psi^h) = 0 & - \\ \nabla_1^3(\nabla_1^3\chi^h + \nabla_1^2\Psi^h) = 0 & = \\ \hline (\nabla_1^2\nabla_1^4 - \nabla_1^3\nabla_1^3)\chi^h = 0 & \end{array} \tag{127}$$

The same can be done for Ψ^h: applying the operator ∇_1^3 to eq. $(126)_1$ and ∇_1^4 to eq. $(126)_2$, then subtracting the first equation from the second one, the result is exactly the same:

$$(\nabla_1^2\nabla_1^4 - \nabla_1^3\nabla_1^3)\Psi^h = 0. \tag{128}$$

Eqs. (127) and (128) are two uncoupled sixth-order differential equations for χ and Ψ.

A final consideration

The mathematical technique for solving such equations is very peculiar: the above equations are transformed into a sequence of six first-order equations, solved successively. Boundary conditions must, of course, be specified too.

All this part, very technical, is left apart here.

In the literature, the problems of plane deformation and of generalized plane stress are often combined and called the plane problem of the theory of elasticity.

We can hence remark that the Lekhnitskii theory is a general frame where generalized plane strain, plane strain and generalized plane stress are special cases.

Nevertheless, the case of plane stress, as defined before, is not comprehended in the Lekhnitskii theory.

The Stroh theory

We will name a Stroh Problem every problem of the elastic equilibrium of an anisotropic body whose displacement field is constrained to satisfy uniquely the condition of plane field

$$\mathbf{u} = \mathbf{u}(x_1, x_2). \tag{129}$$

The same properties are obviously true for $\boldsymbol{\varepsilon}$, and, through the Hooke's law, for $\boldsymbol{\sigma}$:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(x_1, x_2), \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}(x_1, x_2). \tag{130}$$

To remark that a consequence of assumption (129) is that $\varepsilon_{33} = 0$, but not that $\sigma_{33} = 0$.

We can hence notice that all the fields are plane fields in a Stroh problem, but none of them is a plane tensor or vector, because not all the components on x_3 vanish.

The Stroh theory or formalism is the mathematical theory for obtaining a general formulation of the solution to the Stroh Problem.

There are several similarities between the Stroh and the Lekhnitskii theories, but they remain two different approaches, both mathematically speaking than mechanically speaking (the basic assumption is different).

The full development of the Stroh formalism is rather complicate and technical, so it will not be treated here.

Nomenclature for plane problems

We have seen that there are different cases of plane problems: plane strain, plane stress, generalized plane stress etc.

Nevertheless, one can imagine to be in a plane world with only 2 dimensions, and state all the equations in this hypothetic world.

Of course, such a situation can represent different practical situations, like plane strain or plane stress and so on.

In other words, we can continue to work with the classical equations of elasticity in a plane situation, without necessarily specifying in which state actually we are.

In such a case, we will continue to use the customary nomenclature for the Hooke's law:

$$\begin{aligned} \{\sigma\} &= [C]\{\varepsilon\}, \\ \{\varepsilon\} &= [S]\{\sigma\}. \end{aligned} \tag{131}$$

Every time that we will state an equation in a general sense, without the need for specifying to which state it is referred to, we will use the above symbols, namely for the stiffness and compliance tensors.

Whenever the situation is that of plane strain, then we will write

$$
\begin{aligned}
\{\sigma\} &= [C]\{\varepsilon\},\\
\{\varepsilon\} &= [\Sigma]\{\sigma\},
\end{aligned}
\tag{132}
$$

and in case of plane stress or generalized plane stress

$$
\begin{aligned}
\{\sigma\} &= [Q]\{\varepsilon\},\\
\{\varepsilon\} &= [S]\{\sigma\}.
\end{aligned}
\tag{133}
$$

In other words, in case of plane strain and stress we will use the reduced compliance and stiffness tensors respectively, $[\Sigma]$ and $[Q]$.

In all the cases, we omit, for the sake of simplicity, the superscript p for indicating the plane case.